

PRE-TORSORS AND EQUIVALENCES

GABRIELLA BÖHM AND TOMASZ BRZEZIŃSKI

ABSTRACT. Properties of (most general) non-commutative torsors or A - B torsors are analysed. Starting with pre-torsors it is shown that they are equivalent to a certain class of Galois extensions of algebras by corings. It is shown that a class of faithfully flat pre-torsors induces equivalences between categories of comodules of associated corings. It is then proven that A - B torsors correspond to monoidal functors (and, under some additional conditions, equivalences) between categories of comodules of bialgebroids.

1. INTRODUCTION

The notion of an A - B torsor appeared in algebra as a result of a chain of natural generalisations of the notion of a quantum torsor introduced in [16] as a formalisation of the proposal made by Kontsevich in [20, Section 4.2]. As observed in [30], a faithfully flat quantum torsor is the same as a faithfully flat Hopf-(bi)Galois object. To cover the case of a Hopf-Galois extension (rather than just an object), the notion of a B torsor was introduced in [30] (cf. [31, Section 2.8]). Hence B torsors can be understood as Hopf-Galois extensions *without* the explicit mention of a Hopf algebra. Furthermore, a faithfully flat B torsor corresponds not only to a Hopf-Galois extension of the base algebra B on one side, but also to a Galois object (by a B -bialgebroid) on the other. In order to remove this asymmetry, the notion of an A - B torsor was introduced in [17, Chapter 5]. A faithfully flat A - B torsor can be understood as a bi-Galois (i.e. two-sided Galois) extension by bialgebroids.

As observed in [17, Theorem 5.2.10], with every faithfully flat A - B torsor T there are associated two bialgebroids, one over A , the other over B . These bialgebroids coact (freely) on the torsor, making it a bicomodule. Thus A - B torsors are natural objects which can facilitate a description of (monoidal) functors between categories of comodules of bialgebroids. The freeness of coactions of bialgebroids on T (understood as the *Galois condition*) has a very natural geometric interpretation. Recall that a principal bundle over a Lie groupoid is a manifold with a free groupoid action, whose fixed manifold coincides with the base manifold of another Lie groupoid (cf. [23, Section 5.7]). Bialgebroids can be seen as *groupoids* in non-commutative geometry (and for this reason are often referred to as *quantum groupoids*). Thus faithfully flat A - B torsors can be understood as *quantum principal bundles over quantum groupoids*. Since a monoidal functor between the comodule categories of two bialgebroids maps comodule algebras to comodule algebras, such functors play an important role when Galois extensions by different bialgebroids need to be related. This is the case, for example, in the reduction of a quantum principal bundle over a quantum groupoid \mathcal{C} , to a bundle whose structure is described by a quotient of \mathcal{C} .

Date: July 2006; revised April 2007.

1991 *Mathematics Subject Classification.* 16W30; 58B34.

The aim of this paper is to analyse algebraic properties of A - B torsors and their generalisation, termed pre-torsors. We start in Section 2 with recalling some preliminary results about corings, bialgebroids and \times_A -Hopf algebras. The notion of an A - B pre-torsor is introduced in Section 3. It is shown that faithfully flat pre-torsors are in bijective correspondence with faithfully flat coring-Galois extensions. In particular every faithfully flat A - B pre-torsor induces an A -coring \mathcal{C} and a B -coring \mathcal{D} . In the first part of Section 4 it is shown that both corings \mathcal{C} and \mathcal{D} arising from a faithfully flat pre-torsor T can be identified with cotensor products (as bicomodules). If T satisfies some additional conditions (cf. Remark 4.7 (iii)), then it induces an equivalence between the categories of (left) comodules of \mathcal{C} and \mathcal{D} . It is furthermore shown that these additional conditions reduce to a natural faithful flatness assumption provided the entwining maps induced by T are bijective. Section 5 deals with properties specific to A - B torsors. Following [17] we establish a bijective correspondence between faithfully flat A - B torsors and faithfully flat Galois extensions with \times_A - or \times_B -Hopf algebras. In the case when, for an algebra B and an A -bialgebroid \mathcal{C} , the functor from the category of \mathcal{C} -comodules to the category of B - B bimodules, induced by a B^e - \mathcal{C} bicomodule T , preserves colimits, we establish a bijective correspondence between B -ring and \mathcal{C} -comodule algebra structures in T on one hand and lax monoidal structures of the induced functor on the other hand. The question when is this monoidal structure strict is addressed. In particular, it is proven that a faithfully flat A - B torsor which is also faithfully flat as a right B -module induces a strict monoidal functor. Finally a class of A - B torsors inducing monoidal equivalences between categories of comodules is found. The paper is concluded with two appendices. In the first one a \times_B -Hopf algebra corresponding to a torsor coming from a cleft extension by a Hopf algebroid is computed. This turns out to be a generalisation of bialgebroids studied in [11] and [18] (and shown to be mutually isomorphic in [24]). In the second appendix we describe differential structures and differentiable bimodules associated to unital faithfully flat pre-torsors.

2. PRELIMINARIES. CORINGS, BIALGEBROIDS, \times_A -HOPF ALGEBRAS

Throughout the paper, all algebras are over a commutative associative ring k with a unit. Categories of right (resp. left) modules of an algebra A are denoted by \mathcal{M}_A (resp. ${}_A\mathcal{M}$).

Given an algebra A , a coalgebra \mathcal{C} in the monoidal category of A - A bimodules is called an A -*coring*. The coproduct and counit in \mathcal{C} are denoted by $\Delta_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C} \otimes_A \mathcal{C}$ and $\varepsilon_{\mathcal{C}} : \mathcal{C} \rightarrow A$, respectively. If a right A -module T is a right \mathcal{C} -comodule, then (A -linear, coassociative and counital) coaction is denoted by ϱ^T . A left \mathcal{C} -coaction in a left \mathcal{C} -comodule T is denoted by ${}^T\varrho$. The category of right (resp. left) \mathcal{C} -comodules is denoted by $\mathcal{M}^{\mathcal{C}}$ (resp. ${}^{\mathcal{C}}\mathcal{M}$).

Given an A -coring \mathcal{C} and a B -coring \mathcal{D} , a B - A bimodule T is called a \mathcal{D} - \mathcal{C} *bicomodule* if T is a right \mathcal{C} -comodule and left \mathcal{D} -comodule with B - A bilinear coactions which commute in the sense that

$$({}^T\varrho \otimes {}_A\mathcal{C}) \circ \varrho^T = (\mathcal{D} \otimes_B \varrho^T) \circ {}^T\varrho.$$

The category of \mathcal{D} - \mathcal{C} bicomodules is denoted by ${}^{\mathcal{D}}\mathcal{M}^{\mathcal{C}}$. In particular, B is a trivial B -coring with the coproduct and counit given by the identity map. Thus to say that

T is a $B\text{-}\mathcal{C}$ bicomodule is the same as to say that T is a $B\text{-}A$ bimodule with left B -linear right \mathcal{C} -coaction.

For a right \mathcal{C} -comodule T and a left \mathcal{C} -comodule M , the cotensor product is defined as the equaliser of $\varrho^T \otimes_A M, T \otimes_A {}^M\varrho : T \otimes_A M \rightarrow T \otimes_A \mathcal{C} \otimes_A M$ and is denoted by $T \square_{\mathcal{C}} M$. For a $\mathcal{D}\text{-}\mathcal{C}$ bicomodule T and a left \mathcal{C} -comodule M , the cotensor product $T \square_{\mathcal{C}} M$ is a left \mathcal{D} -comodule provided that $T \square_{\mathcal{C}} M$ is a $\mathcal{D} \otimes_B \mathcal{D}$ -pure equaliser in $_B\mathcal{M}$ (cf. [10, 22.3, erratum]). If this purity condition holds for every left \mathcal{C} -comodule M , then $T \square_{\mathcal{C}} \bullet$ defines a functor ${}^{\mathcal{C}}\mathcal{M} \rightarrow {}^{\mathcal{D}}\mathcal{M}$. The cotensor functor induced by any $\mathcal{D}\text{-}\mathcal{C}$ bicomodule T exists in particular if \mathcal{D} is a flat right B -module.

For more information on corings, the reader is referred to [10].

An algebra T in the monoidal category of $A\text{-}A$ bimodules is called an *A-ring*. An *A-ring* T is equivalent to a k -algebra T and a k -algebra map $\eta_T : A \rightarrow T$ (which serves as the *unit map* for the *A-ring* T). The unit element in the k -algebra T is denoted by 1_T . The product in T both as a map $T \otimes_k T \rightarrow T$ and $T \otimes_A T \rightarrow T$ is denoted by μ_T .

A triple (T, \mathcal{C}, ψ) , where T is an *A-ring*, \mathcal{C} is an *A-coring* and $\psi : \mathcal{C} \otimes_A T \rightarrow T \otimes_A \mathcal{C}$ is an $A\text{-}A$ bilinear map such that

$$\begin{aligned} \psi \circ (\mathcal{C} \otimes_A \mu_T) &= (\mu_T \otimes_A \mathcal{C}) \circ (T \otimes_A \psi) \circ (\psi \otimes_A T), & \psi \circ (\mathcal{C} \otimes_A \eta_T) &= \eta_T \otimes_A \mathcal{C}, \\ (T \otimes_A \Delta_{\mathcal{C}}) \circ \psi &= (\psi \otimes_A \mathcal{C}) \circ (\mathcal{C} \otimes_A \psi) \circ (\Delta_{\mathcal{C}} \otimes_A T), & (T \otimes_A \varepsilon_{\mathcal{C}}) \circ \psi &= \varepsilon_{\mathcal{C}} \otimes_A T, \end{aligned}$$

is called a *right entwining structure over A*. A *right entwined module* is a right \mathcal{C} -comodule and right T -module M , with multiplication $\varrho_M : M \otimes_A T \rightarrow M$, such that

$$\varrho^M \circ \varrho_M = (\varrho_M \otimes_A \mathcal{C}) \circ (M \otimes_A \psi) \circ (\varrho^M \otimes_A T).$$

The morphisms of entwined modules are right A -linear, right \mathcal{C} -colinear maps. If T itself is an entwined module with multiplication μ_T and coaction ϱ^T , then, for any entwined module M , the *coinvariants* are defined by

$$M^{co\mathcal{C}} := \{m \in M \mid \varrho^M(m) = m\varrho^T(1_T)\}.$$

In particular, to any *A-coring* \mathcal{C} one can associate a *trivial right entwining structure* $(A, \mathcal{C}, \mathcal{C})$ (i.e. the entwining map ψ is the identity map on \mathcal{C}). Entwined modules in this case are simply right comodules of \mathcal{C} . Thus to say that A is an entwined module in this case is the same as to say that \mathcal{C} has a group-like element g , and $\varrho^A(1_A) = 1_A \otimes_A g$. Consequently the coinvariants of a right \mathcal{C} -comodule N coincide with $\{n \in N \mid \varrho^N(n) = n \otimes_A g\}$.

Every right entwining structure (T, \mathcal{C}, ψ) over A gives rise to a T -coring $T \otimes_A \mathcal{C}$. Entwined modules can be identified with right comodules of this T -coring.

Left entwining structures are defined symmetrically. In particular if ψ in a right entwining structure (T, \mathcal{C}, ψ) is bijective, then $(T, \mathcal{C}, \psi^{-1})$ is a left entwining structure.

Let \mathcal{C} be an *A-coring* and T be an *A-ring* which is also a right \mathcal{C} -comodule (with right A -module structure given by the unit map). Define $B = \{b \in T \mid \forall t \in T \varrho^T(bt) = b\varrho^T(t)\}$. We say that T is a *right \mathcal{C} -Galois extension* of B if the canonical Galois map

$$\text{can} : T \otimes_B T \rightarrow T \otimes_A \mathcal{C}, \quad t \otimes_B t' \mapsto t\varrho^T(t'),$$

is bijective. The restricted inverse of can , $\chi : \mathcal{C} \rightarrow T \otimes_B T$, $c \mapsto \text{can}^{-1}(1_T \otimes_A c)$, is called the *translation map*. If T is a left \mathcal{C} -comodule, one defines a *left \mathcal{C} -Galois extension* in analogous way, using the left coaction to define the *left Galois map*. When

dealing with both right and left \mathcal{C} -Galois extensions we write $\text{can}_{\mathcal{C}}$ for the right Galois map and ${}_c\text{can}$ for the left Galois map.

Every right (resp. left) \mathcal{C} -Galois extension $B \subseteq T$ gives rise to a right (resp. left) entwining structure (T, \mathcal{C}, ψ) with $\psi : c \otimes_A t \mapsto \text{can}_{\mathcal{C}}(\text{can}_{\mathcal{C}}^{-1}(1_T \otimes_A c) \cdot t)$ (resp. $\psi : t \otimes_A c \mapsto {}_c\text{can}(t \cdot {}_c\text{can}^{-1}(c \otimes_A 1_T))$), for which T is an entwined module. One then easily shows that $B = T^{co\mathcal{C}}$.

An A -coring \mathcal{C} is a *Galois coring* if A itself is a \mathcal{C} -Galois extension of the coinvariant subalgebra $\{b \in A \mid \forall a \in A \quad \varrho^A(ba) = b\varrho^A(a)\}$ (which is the commutant in the A - A bimodule \mathcal{C} of the grouplike element g determining the \mathcal{C} -coaction in A). The T -coring $T \otimes_A \mathcal{C}$, arising from an entwining structure determined by a \mathcal{C} -Galois extension $B \subseteq T$, is a Galois coring (with grouplike element $\varrho^T(1_T)$). The coinvariants of T with respect to \mathcal{C} and $T \otimes_A \mathcal{C}$ are the same.

For a T -coring \mathcal{E} with a grouplike element, there exist adjoint functors

$$(2.1) \quad (\bullet)^{co\mathcal{E}} : \mathcal{M}^{\mathcal{E}} \rightarrow \mathcal{M}_B \quad \text{and} \quad \bullet \otimes_B T : \mathcal{M}_B \rightarrow \mathcal{M}^{\mathcal{E}},$$

(cf. [10, 28.8]) where $B := T^{co\mathcal{E}}$ is the coinvariant subalgebra. By the Galois Coring Structure Theorem [10, 28.19 (2) (a) \Rightarrow (c)], the functors (2.1) are inverse equivalences provided that \mathcal{E} is a Galois coring and T is a faithfully flat left B -module.

Lemma 2.1. *Let \mathcal{E} be a Galois coring over an algebra T and let A be an algebra. Assume that T is a faithfully flat left module for its \mathcal{E} -coinvariant subalgebra B . For any A - \mathcal{E} bicomodule M , $M^{co\mathcal{E}}$ is a pure equaliser in ${}_A\mathcal{M}$.*

Proof. For any right A -module N , $N \otimes_A M$ inherits a right \mathcal{E} -comodule structure of M . Applying twice the Galois Coring Structure Theorem [10, 28.19 (2) (a) \Rightarrow (c)], one concludes that

$$(N \otimes_A M)^{co\mathcal{E}} \otimes_B T \cong N \otimes_A M \cong N \otimes_A M^{co\mathcal{E}} \otimes_B T.$$

Hence the claim follows by the faithful flatness of T as a left B -module. \square

By an analogy to rings, a B -coring \mathcal{D} is termed a *right extension* of an A -coring \mathcal{C} if the forgetful functor $\mathcal{M}^{\mathcal{C}} \rightarrow \mathcal{M}_k$ factorises through a k -linear functor $\mathcal{M}^{\mathcal{C}} \rightarrow \mathcal{M}^{\mathcal{D}}$ and the forgetful functor $\mathcal{M}^{\mathcal{D}} \rightarrow \mathcal{M}_k$. In [8, Theorem 2.6] this definition was shown to be equivalent to the existence of a right \mathcal{D} -coaction on \mathcal{C} , which is left colinear with respect to the left regular comodule structure of \mathcal{C} . Left extensions of corings are defined symmetrically, in terms of the categories of left comodules.

Extensions of corings over the same base algebra are related to coring morphisms as follows.¹

Lemma 2.2. *For two A -corings \mathcal{C} and $\tilde{\mathcal{C}}$, denote the forgetful functors by $F : \mathcal{M}^{\mathcal{C}} \rightarrow \mathcal{M}_A$ and $\tilde{F} : \mathcal{M}^{\tilde{\mathcal{C}}} \rightarrow \mathcal{M}_A$. Then the following assertions are equivalent.*

- (i) *There is a k -linear functor $U : \mathcal{M}^{\mathcal{C}} \rightarrow \mathcal{M}^{\tilde{\mathcal{C}}}$ such that $F = \tilde{F} \circ U$.*
- (ii) *Considering the A -bimodule \mathcal{C} as a left \mathcal{C} -comodule via the coproduct, there is a right $\tilde{\mathcal{C}}$ -coaction $\tilde{\varrho} : \mathcal{C} \rightarrow \mathcal{C} \otimes_A \tilde{\mathcal{C}}$, making \mathcal{C} a \mathcal{C} - $\tilde{\mathcal{C}}$ bicomodule.*
- (iii) *There is a homomorphism of A -corings $\kappa : \mathcal{C} \rightarrow \tilde{\mathcal{C}}$.*

¹Lemma 2.2 and new formulation of Lemma 3.7 replace incorrect statements on page 548 and in Lemma 3.7 of the published version of this paper: [G. Böhm & T. Brzeziński, *Pre-torsors and equivalences*, J. Algebra 317 (2007) 544–580].

Proof. (i) \Rightarrow (ii) By property (i), there is a right $\tilde{\mathcal{C}}$ -coaction $\tilde{\varrho}$ on the right A -module \mathcal{C} . Since under assumption (i) $\tilde{\mathcal{C}}$ is a right extension of \mathcal{C} , $\tilde{\varrho}$ is a left \mathcal{C} -comodule map by [8, Theorem 2.6].

- (ii) \Rightarrow (iii) The map κ is constructed as $\kappa := (\epsilon_{\mathcal{C}} \otimes_A \tilde{\mathcal{C}}) \circ \tilde{\varrho}$.
- (iii) \Rightarrow (i) The functor U is given by the *corestriction functor* along κ , cf. [10, 22.11]. \square

A symmetrical statement holds for the categories of left (co)modules. Lemma 2.2 (ii) asserts that the A -module structures of \mathcal{C} , as an A -coring on one hand and as a $\tilde{\mathcal{C}}$ -comodule on the other, are the same. Note that this property allows the construction in the proof of the implication (ii) \Rightarrow (iii) in Lemma 2.2 to yield a well defined and right A -linear map κ .

A right bialgebroid over A [33], [21] is a quintuple $\mathcal{C} = (\mathcal{C}, s, t, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$. Here $s, t : A \rightarrow \mathcal{C}$ are k -linear maps, \mathcal{C} is an $A \otimes_k A^{op}$ -ring with the unit map $\mu_{\mathcal{C}} \circ (s \otimes_k t)$ (thus, in particular, s is an algebra and t is an anti-algebra morphism), and $(\mathcal{C}, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$ is an A -coring. The bimodule structure of this A -coring is given by

$$aca' = cs(a')t(a), \quad \text{for all } a, a' \in A, c \in \mathcal{C}.$$

The range of the coproduct is required to be in the *Takeuchi product*

$$\mathcal{C} \times_A \mathcal{C} := \left\{ \sum_i c_i \otimes_A c'_i \in \mathcal{C} \otimes_A \mathcal{C} \mid \forall a \in A, \sum_i s(a)c_i \otimes_A c'_i = \sum_i c_i \otimes_A t(a)c'_i \right\},$$

which is an algebra by factorwise multiplication, and the (corestriction of the) coproduct is required to be an algebra map. The counit satisfies the following conditions

$$\varepsilon_{\mathcal{C}}(1_{\mathcal{C}}) = 1_A, \quad \varepsilon_{\mathcal{C}}(s(\varepsilon_{\mathcal{C}}(c))c') = \varepsilon_{\mathcal{C}}(t(\varepsilon_{\mathcal{C}}(c))c') = \varepsilon_{\mathcal{C}}(cc'),$$

for all $c, c' \in \mathcal{C}$. The map s is called a *source map*, and t is known as a *target map*. Left bialgebroids are defined analogously in terms of multiplications by the source and target maps on the left. For more details we refer to [19].

The category of right comodules of a right A -bialgebroid $(\mathcal{C}, s, t, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$ is a monoidal category with a strict monoidal functor to the category of A - A bimodules [28, Proposition 5.6]. Explicitly, take a right comodule M of \mathcal{C} (i.e. of the A -coring $(\mathcal{C}, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$), and define the left A -multiplication on M in terms of the coaction $\varrho^M : M \rightarrow M \otimes_A \mathcal{C}$, $m \mapsto m^{[0]} \otimes_A m^{[1]}$,

$$am = m^{[0]}\varepsilon_{\mathcal{C}}(t(a)m^{[1]}) = m^{[0]}\varepsilon_{\mathcal{C}}(s(a)m^{[1]}), \quad \text{for all } a \in A, m \in M.$$

This is a unique left multiplication which guarantees that the image of the coaction ϱ^M is a subset of the Takeuchi product

$$M \times_A \mathcal{C} := \left\{ \sum_i m_i \otimes_A c_i \in M \otimes_A \mathcal{C} \mid \forall a \in A, \sum_i am_i \otimes_A c_i = \sum_i m_i \otimes_A t(a)c_i \right\}.$$

In addition it equips M with an A - A bimodule structure such that every \mathcal{C} -colinear map becomes A - A bilinear. For any right \mathcal{C} -comodules M and N , the right A -multiplication and the right \mathcal{C} -coaction on the tensor product $M \otimes_A N$ are defined by

$$(m \otimes_A n) \cdot a = m \otimes_A na, \quad (m \otimes_A n)^{[0]} \otimes_A (m \otimes_A n)^{[1]} = (m^{[0]} \otimes_A n^{[0]}) \otimes_A m^{[1]}n^{[1]},$$

for all $a \in A$, $m \otimes_A n \in M \otimes_A N$. The monoidal unit is A , with the regular module structure and coaction given by the source map. In a symmetric way, also the left comodules of a right A -bialgebroid $(\mathcal{C}, s, t, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$ form a monoidal category with

respect to the A^{op} -module tensor product. Any left \mathcal{C} -comodule M , with coaction ${}^M\varrho : M \rightarrow \mathcal{C} \otimes_A M$, $m \mapsto m^{[-1]} \otimes_A m^{[0]}$, is equipped with a right A -module structure

$$ma = \varepsilon_{\mathcal{C}}(s(a)m^{[-1]})m^{[0]} = \varepsilon_{\mathcal{C}}(t(a)m^{[-1]})m^{[0]}, \quad \text{for all } a \in A, m \in M.$$

With this definition, the range of ${}^M\varrho$ is in the Takeuchi product

$$\mathcal{C} \times_A M := \{ \sum_i c_i \otimes_A m_i \in \mathcal{C} \otimes_A M \mid \forall a \in A, \sum_i c_i \otimes_A m_i a = \sum_i s(a)c_i \otimes_A m_i \},$$

and M is an A - A (equivalently, A^{op} - A^{op}) bimodule such that any comodule map is bilinear. For any two left \mathcal{C} -comodules M and N , with coactions $m \mapsto m^{[-1]} \otimes_A m^{[0]}$ and $n \mapsto n^{[-1]} \otimes_A n^{[0]}$, respectively, the left A -action and left \mathcal{C} -coaction are

$$a(m \otimes_{A^{op}} n) = m \otimes_{A^{op}} an, \quad (m \otimes_{A^{op}} n)^{[-1]} \otimes_A (m \otimes_{A^{op}} n)^{[0]} = m^{[-1]}n^{[-1]} \otimes_A (m^{[0]} \otimes_{A^{op}} n^{[0]}),$$

for $a \in A$ and $m \otimes_{A^{op}} n \in M \otimes_{A^{op}} N$. The monoidal unit is A^{op} , with the left regular A -module structure and coaction given by the target map.

A right *comodule algebra* T for a right bialgebroid $(\mathcal{C}, s, t, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$ over A is an algebra in the monoidal category of right comodules for $(\mathcal{C}, s, t, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$. By the existence of a strict monoidal forgetful functor from this comodule category to the category of A - A bimodules, T is in particular an A -ring. The A -ring T and the A -coring $(\mathcal{C}, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$ are canonically entwined. A right *Galois extension* by a right bialgebroid $(\mathcal{C}, s, t, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$ over A is a right comodule algebra T , which is a Galois extension of its coinvariant subalgebra by the A -coring $(\mathcal{C}, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$. The range of the translation map in this case is contained in the center of the B - B bimodule $T \otimes_B T$.

Note that the definition of the left A -multiplication on a right \mathcal{C} -comodule T implies that, for any left \mathcal{C} -comodule M , the cotensor product $T \square_{\mathcal{C}} M$ is contained in the A -centraliser of $T \otimes_A M$. That is, denote by $t \mapsto t^{[0]} \otimes_A t^{[1]}$ and $m \mapsto m^{[-1]} \otimes_A m^{[0]}$ the right coaction in T and the left coaction in M , respectively. For $\sum_i t_i \otimes_A m_i \in T \square_{\mathcal{C}} M$ and $a \in A$,

$$\begin{aligned} a(\sum_i t_i \otimes_A m_i) &= \sum_i t_i^{[0]} \varepsilon_{\mathcal{C}}(s(a)t_i^{[1]}) \otimes_A m_i \\ &= \sum_i t_i \otimes_A \varepsilon_{\mathcal{C}}(s(a)m_i^{[-1]})m_i^{[0]} = (\sum_i t_i \otimes_A m_i)a. \end{aligned}$$

The second equality follows by the definition of the cotensor product as an equaliser.

Given a right bialgebroid $(\mathcal{C}, s, t, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$ over A , one can define the Galois map corresponding to the right regular \mathcal{C} -comodule, with coinvariants $t(A) \cong A^{op}$,

$$(2.2) \quad \theta : \mathcal{C} \otimes_{A^{op}} \mathcal{C} \rightarrow \mathcal{C} \otimes_A \mathcal{C}, \quad c \otimes_A c' \mapsto c\Delta_{\mathcal{C}}(c').$$

The map θ satisfies the pentagon identity

$$(2.3) \quad (\theta \otimes \mathcal{C}) \circ (\mathcal{C} \otimes_{A^{op}} \theta) = (\mathcal{C} \otimes_A \theta) \circ \theta_{13} \circ (\theta \otimes_{A^{op}} \mathcal{C}),$$

where $\theta_{13} : (\mathcal{C}_\bullet \otimes_A \mathcal{C}) \otimes_{A^{op}} \mathcal{C} \rightarrow \mathcal{C} \otimes_A (\mathcal{C} \otimes_{A^{op}} \bullet \mathcal{C})$ is the map defined as the nontrivial action of θ on the first and the third factors. Following [29], a right bialgebroid $(\mathcal{C}, s, t, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$ over A is called a \times_A -*Hopf algebra* iff the Galois map θ is bijective. Similarly, for left A -bialgebroids, one considers the left Galois map $c \otimes_{A^{op}} c' \mapsto \Delta_{\mathcal{C}}(c)c'$ and defines a left \times_A -Hopf algebra by requiring this map be bijective. The translation map $\theta^{-1}(1_{\mathcal{C}} \otimes_A \bullet)$, corresponding to (2.2), is an algebra map from \mathcal{C} to the center

of the A^{op} - A^{op} bimodule $\mathcal{C} \otimes_{A^{op}} \mathcal{C}$, where all A^{op} -module structures are given by the target map and the algebra structure is inherited from $\mathcal{C}^{op} \otimes_k \mathcal{C}$. Furthermore,

$$(2.4) \quad \theta^{-1}(1_{\mathcal{C}} \otimes t(a)) = s(a) \underset{A^{op}}{\otimes} 1_{\mathcal{C}} \quad \text{and} \quad \theta^{-1}(1_{\mathcal{C}} \otimes s(a)) = 1_{\mathcal{C}} \underset{A^{op}}{\otimes} s(a),$$

for all $a \in A$.

Let $(\mathcal{C}, s, t, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$ be a right bialgebroid over A which admits a Galois extension $B \subseteq T$ such that T is a faithfully flat right A -module. By [17, Lemma 4.1.21], the existence of such an extension implies that $(\mathcal{C}, s, t, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$ is a \times_A -Hopf algebra.

3. PRE-TORSORS, CORING-GALOIS EXTENSIONS AND BI-GALOIS OBJECTS

One of the most striking properties of A - B torsors is the observation made in [17, Theorem 5.2.10] that, in the faithfully flat case, they correspond to Galois extensions by bialgebroids. Guided by the non-commutative geometry experience, where the notion of a Hopf-Galois extension is not flexible enough to describe examples of quantum principal bundles, one can also envisage that the notion of an A - B torsor might be too strict to deal with bundles over quantum groupoids (cf. Example 3.9). In this section we introduce the notion of an A - B *pre-torsor*. In the faithfully flat case we show that this notion is equivalent to a certain class of coring-Galois extensions.

Definition 3.1. Let $\alpha : A \rightarrow T$ and $\beta : B \rightarrow T$ be k -algebra maps. View T as an A - B bimodule and a B - A bimodule via the maps α and β . We say that T is an A - B *pre-torsor* if there exists a B - A bimodule map

$$\tau : T \rightarrow T \underset{A}{\otimes} T \underset{B}{\otimes} T, \quad t \mapsto t^{\langle 1 \rangle} \underset{A}{\otimes} t^{\langle 2 \rangle} \underset{B}{\otimes} t^{\langle 3 \rangle},$$

(summation understood) such that

- (a) $(\mu_T \underset{B}{\otimes} T) \circ \tau = \beta \underset{B}{\otimes} T$;
- (b) $(T \underset{A}{\otimes} \mu_T) \circ \tau = T \underset{A}{\otimes} \alpha$;
- (c) $(\tau \underset{A}{\otimes} T \underset{B}{\otimes} T) \circ \tau = (T \underset{A}{\otimes} T \underset{B}{\otimes} \tau) \circ \tau$,

where μ_T denotes the quotients of the product in T to appropriate tensor products (over A or B).

An A - B pre-torsor is said to be *faithfully flat*, if it is faithfully flat as a right A -module and left B -module. (Note that in this case the unit maps α and β are injective.)

Following the observation made in [4, Remark 2.4] we propose the following

Definition 3.2. Given an A -coring \mathcal{C} and a B -coring \mathcal{D} , an algebra T is called a \mathcal{C} - \mathcal{D} *bi-Galois object* if T is a \mathcal{D} - \mathcal{C} bicomodule, and T is a right \mathcal{C} -Galois extension of B and a left \mathcal{D} -Galois extension of A . A \mathcal{C} - \mathcal{D} bi-Galois object T is said to be *faithfully flat*, if T is faithfully flat as a left B -module and a right A -module.

Since corings can be understood as the algebraic structure underlying quantum (Lie) groupoids, the \mathcal{C} -Galois extension $B \subseteq T$ can be understood as the dual to a free action of a groupoid on a manifold. With this interpretation in mind a (faithfully flat) \mathcal{C} - \mathcal{D} bi-Galois object T can be seen as a non-commutative version of a groupoid principal bundle over a groupoid (cf. [23, Section 5.7]).

The main result of this section, which includes the pre-torsor version of [17, Theorem 5.2.10], is contained in Theorem 3.4 below.

Remark 3.3. The basic difference between [17, Theorem 5.2.10] and Theorem 3.4 is that the former one is formulated for A - B torsors, while the latter one deals with pre-torsors, i.e. without multiplicative structure of the torsor map. Furthermore, [17, Theorem 5.2.10] deals with the analogues of the implications $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (iii)$ in Theorem 3.4 not with equivalences. Also, in [17, Theorem 5.2.10] a torsor T is assumed to be faithfully flat both as a left and right module, for both base algebras A and B . In contrast, we assume only the faithful flatness of a pre-torsor T as a right A -module and as a left B -module, cf. Definition 3.1. Thanks to Lemma 3.6, these two faithful flatness assumptions are enough to conclude the claim.

Theorem 3.4. *There is a bijective correspondence between the following sets of data:*

- (i) *faithfully flat A - B pre-torsors T ;*
- (ii) *A -corings \mathcal{C} and left faithfully flat right \mathcal{C} -Galois extensions $B \subseteq T$, such that T is a right faithfully flat A -ring;*
- (iii) *B -corings \mathcal{D} and right faithfully flat left \mathcal{D} -Galois extensions $A \subseteq T$, such that T is a left faithfully flat B -ring.*

Furthermore, a faithfully flat A - B pre-torsor T is a faithfully flat \mathcal{C} - \mathcal{D} bi-Galois object with \mathcal{C} and \mathcal{D} as in parts (ii) and (iii).

Proof. We will prove the equivalence of statements (i) and (ii). The equivalence of (i) with (iii) will then follow by the symmetric nature of the notion of an A - B pre-torsor. The implication $(ii) \Rightarrow (i)$ is a consequence of the following

Lemma 3.5. *Let \mathcal{C} be an A -coring and T an A -ring. If $B \subseteq T$ is a right \mathcal{C} -Galois extension, then T is an A - B pre-torsor with the structure map*

$$\tau := (T \otimes_A \chi) \circ \varrho^T : T \rightarrow T \otimes_A T \otimes_B T,$$

where $\varrho^T : T \rightarrow T \otimes_A \mathcal{C}$ is the coaction and $\chi : \mathcal{C} \rightarrow T \otimes_B T$ is the translation map.

Proof. Since ϱ^T is a B - A bimodule map and χ is right A -linear, the map τ is B - A bilinear. Note that $(\mu_T \otimes_B T) \circ \tau = (\text{can})^{-1} \circ \varrho^T$, and $\varrho^T = \text{can} \circ (\beta \otimes_B T)$, where β is the obvious inclusion $B \subseteq T$. This implies that τ satisfies condition (a) in Definition 3.1. The condition (b) follows by noting that $\alpha \circ \varepsilon_{\mathcal{C}} = \mu_T \circ \chi$, where α is the unit map $A \rightarrow T$. A simple calculation which uses the coassociativity of ϱ^T and right \mathcal{C} -colinearity of χ confirms that τ satisfies property (c). \square

Now assume that T is a faithfully flat A - B pre-torsor with the structure map τ . The proof of the converse implication $(i) \Rightarrow (ii)$ starts with the following

Lemma 3.6. *The equaliser:*

$$(3.1) \quad \mathcal{C} \longrightarrow T \otimes_B T \xrightarrow{\begin{array}{c} (\mu_T \otimes_A T \otimes_B T) \circ (T \otimes_B \tau) \\ \alpha \otimes_A T \otimes_B T \end{array}} T \otimes_A T \otimes_B T$$

is pure in \mathbf{M}_A .

*Proof.*² Let $\omega = (\mu_T \otimes_A T \otimes_B T) \circ (T \otimes_B \tau) - \alpha \otimes_A T \otimes_B T$. For any left A -module N , define a map $\varphi_N : T \otimes_A \ker(\omega \otimes_A N) \rightarrow T \otimes_B T \otimes_A N$, as the restriction of $\mu_T \otimes_B T \otimes_A N$.

²The referee pointed out an alternative proof of Lemma 3.6. It is based on the observation that the equaliser, obtained by tensoring (3.1) with the faithfully flat right A -module T on the left, is contractible hence cosplit.

Consider also the map $\tilde{\varphi}_N := (\mu_T \otimes_A T \otimes_B T \otimes_A N) \circ (T \otimes_B \tau \otimes_A N)$. Using properties Definition 3.1 (b) and (c) one easily finds that $(T \otimes_A \omega \otimes_A N) \circ \tilde{\varphi}_N = 0$, hence, in view of the flatness of T as a right A -module, $\text{Im } \tilde{\varphi}_N \subseteq T \otimes_A \ker(\omega \otimes_A N)$. The equality $\varphi_N \circ \tilde{\varphi}_N = T \otimes_B T \otimes_A N$ follows by Definition 3.1 (a), while the definition of ω implies that the composition $\tilde{\varphi}_N \circ \varphi_N$ is the identity too. Thus φ_N is a left T -module isomorphism (which is T - A -linear if N is an A -bimodule) and hence the composition

$$T \otimes_A \ker \omega \otimes_A N \xrightarrow{\varphi_A \otimes_A N} T \otimes_B T \otimes_A N \xrightarrow{\varphi_N^{-1}} T \otimes_A \ker(\omega \otimes_A N),$$

which is simply the obvious map, is an isomorphism too. By the faithful flatness of T , this yields the required isomorphism $\ker \omega \otimes_A N \cong \ker(\omega \otimes_A N)$. \square

The equaliser \mathcal{C} in Lemma 3.6 is given in terms of A -bimodule maps, hence it is an A -bimodule with the A -multiplications given through α ,

$$a \left(\sum_i t_i \otimes_B u_i \right) a' := \sum_i \alpha(a) t_i \otimes_B u_i \alpha(a').$$

Define an A -bimodule map

$$\Delta_{\mathcal{C}} : \mathcal{C} \rightarrow T \otimes_B T \otimes_A T \otimes_B T, \quad \Delta_{\mathcal{C}} = T \otimes_B \tau.$$

Since $\tau = \tilde{\varphi}_A \circ (\beta \otimes_B T)$ (cf. the proof of Lemma 3.6), the range of τ is contained in $T \otimes_A \mathcal{C}$, hence $\Delta_{\mathcal{C}}(\mathcal{C}) \subseteq T \otimes_B T \otimes_A \mathcal{C}$. Using the definition of \mathcal{C} as the kernel of the map ω (cf. the proof of Lemma 3.6) and property (c) in Definition 3.1, one immediately finds that $\text{Im } \Delta_{\mathcal{C}} \subseteq \ker(\omega \otimes_A \mathcal{C})$, hence $\Delta_{\mathcal{C}}(\mathcal{C}) \subseteq \mathcal{C} \otimes_A \mathcal{C}$ by Lemma 3.6.

The map $\Delta_{\mathcal{C}}$ is coassociative by Definition 3.1 (c). Note that property (b) in Definition 3.1 implies that $(T \otimes_A \mu_T) \circ \omega = \mu_T \otimes_A \alpha - \alpha \otimes_A \mu_T$. This means that, for all $c \in \mathcal{C}$, $\mu_T(c) \otimes_A 1_T = 1_T \otimes_A \mu_T(c)$. In view of the faithful flatness of the right A -module T , this implies that there is an A - A bimodule map

$$\varepsilon_{\mathcal{C}} : \mathcal{C} \rightarrow A, \quad \sum_i t_i \otimes_B u_i \mapsto \sum_i t_i u_i.$$

The map $\varepsilon_{\mathcal{C}}$ is a counit for $\Delta_{\mathcal{C}}$ by properties (a) and (b) in Definition 3.1.

The torsor map $\tau : T \rightarrow T \otimes_A \mathcal{C}$ is a right \mathcal{C} -coaction of \mathcal{C} on T , which is coassociative by property (c) in Definition 3.1, and counital by Definition 3.1 (b). By definition, τ is left B -linear. Conversely, if $b \in T$ is such that, for all $t \in T$, $\tau(bt) = b\tau(t)$, then property (a) in Definition 3.1 implies that $1_T \otimes_B bt = b \otimes_B t$, hence $b \in \text{Im } \beta$ by the faithful flatness of T as a left B -module. Finally note that the canonical map $T \otimes_B T \rightarrow T \otimes_A \mathcal{C}$ is the same as the bijective map $\tilde{\varphi}_A$ constructed in the proof of Lemma 3.6. Therefore, $B \subseteq T$ is a right \mathcal{C} -Galois extension as required.

Thus we have established a correspondence between faithfully flat A - B pre-torsors and faithfully flat right \mathcal{C} -Galois extensions. There is a symmetric correspondence between faithfully flat A - B pre-torsors T and faithfully flat left \mathcal{D} -Galois extensions $A \subseteq T$, for the B -coring $\mathcal{D} = \ker((T \otimes_A T \otimes_B \mu_T) \circ (\tau \otimes_A T) - T \otimes_A T \otimes_B \beta)$. Both the right \mathcal{C} -coaction and the left \mathcal{D} -coaction in T are given by the torsor map τ , hence T is a \mathcal{D} - \mathcal{C} bicomodule by property (c) in Definition 3.1.

It remains to prove that the established correspondence between faithfully flat A - B pre-torsors and faithfully flat coring-Galois extensions is bijective. This is a consequence of the following

Lemma 3.7. *Let T be a faithfully flat A - B pre-torsor and \mathcal{C} and \mathcal{D} be the associated A - and B -corings, respectively. Let $\tilde{\mathcal{C}}$ be an A -coring for which T is a \mathcal{D} - $\tilde{\mathcal{C}}$ bicomodule. Then the following hold:*

- (1) *If the right A -action of T as a right $\tilde{\mathcal{C}}$ -comodule is determined by its A -ring structure, then there is a homomorphism of A -corings $\kappa : \mathcal{C} \rightarrow \tilde{\mathcal{C}}$.*
- (2) *$B \subseteq T$ is a right $\tilde{\mathcal{C}}$ -Galois extension if and only if the coring homomorphism κ in part (1) is an isomorphism.*

Proof. (1) In view of Lemma 2.2 (ii) \Rightarrow (iii), we need to construct a right $\tilde{\mathcal{C}}$ -coaction on the right A -module \mathcal{C} which is left colinear with respect to the regular comodule structure of \mathcal{C} . We claim that such a coaction is given in terms of the $\tilde{\mathcal{C}}$ coaction in T , $t \mapsto t^{[0]} \otimes_A t^{[1]}$, as

$$(3.2) \quad \mathcal{C} \rightarrow \mathcal{C} \otimes_A \tilde{\mathcal{C}}, \quad \sum_i u_i \otimes_B v_i \mapsto \sum_i u_i \otimes_B v_i^{[0]} \otimes_A v_i^{[1]}.$$

Applying the map $\omega \otimes_A \tilde{\mathcal{C}}$ in Lemma 3.6 to the range of (3.2), and using the \mathcal{D} - $\tilde{\mathcal{C}}$ bicomodule property of T together with the characterisation of \mathcal{C} as the kernel of ω , we conclude that the range of the map (3.2) is in $\ker(\omega \otimes_A \tilde{\mathcal{C}})$. Hence by Lemma 3.6 it is in $\mathcal{C} \otimes_A \tilde{\mathcal{C}}$, as required. Counitality and coassociativity of the coaction (3.2) hold by the counitality and coassociativity of the $\tilde{\mathcal{C}}$ -coaction in T . Its left \mathcal{C} -colinearity follows immediately by the \mathcal{D} - $\tilde{\mathcal{C}}$ bicomodule property of T . Since the right A -multiplication in T is determined by the A -ring structure of T , the assumptions of Lemma 2.2 (ii) are satisfied, and hence

$$\kappa : \mathcal{C} \rightarrow \tilde{\mathcal{C}}, \quad \sum_i u_i \otimes_B v_i \mapsto \sum_i (u_i v_i^{[0]}) v_i^{[1]}$$

is a homomorphism of A -corings.

(2) Recall that in a $\tilde{\mathcal{C}}$ -Galois extension $B \subseteq T$, T is an A -ring and a $\tilde{\mathcal{C}}$ -comodule via the same right A -actions, so the homomorphism κ in part (1) can be constructed. It follows by Definition 3.1 (a) that $(T \otimes_A \kappa) \circ \text{can}_{\mathcal{C}}$ is equal to the $\tilde{\mathcal{C}}$ -canonical map $\text{can}_{\tilde{\mathcal{C}}}$. Hence if κ is an isomorphism of corings, then a \mathcal{C} -Galois extension $B \subseteq T$ is $\tilde{\mathcal{C}}$ -Galois. Conversely, if $B \subseteq T$ is a right $\tilde{\mathcal{C}}$ -Galois extension, i.e. $\text{can}_{\tilde{\mathcal{C}}}$ is bijective, then the bijectivity of κ follows by the bijectivity of $\text{can}_{\mathcal{C}}$ and the faithful flatness of T as a right A -module. \square

This completes the proof of Theorem 3.4. \square

We conclude this section with a number of examples of A - B pre-torsors.

Example 3.8. The simplest examples of Galois extensions by corings are Galois corings. Indeed, for a Galois T -coring \mathcal{C} , the base algebra T is a \mathcal{C} -Galois extension of the coinvariant subalgebra $B := T^{\text{co}\mathcal{C}}$. By Lemma 3.5, to this \mathcal{C} -Galois extension there corresponds a T - B pre-torsor (T, τ) , with

$$\tau : T \rightarrow T \otimes_T T \otimes_B T \cong T \otimes_B T, \quad t \mapsto 1_T \otimes t.$$

Example 3.9. Examples of Galois extensions by a coring can be obtained by generalising the construction of quantum homogenous spaces due to Schneider [32] [25] from Hopf algebras to \times_A -Hopf algebras.

For an algebra A , consider a right \times_A -Hopf algebra \mathcal{C} , with source map s , target map t , coproduct $\Delta_{\mathcal{C}}$, counit $\varepsilon_{\mathcal{C}}$ and bijective canonical map

$$\theta : \mathcal{C} \otimes_{A^{op}} \mathcal{C} \rightarrow \mathcal{C} \otimes_A \mathcal{C}, \quad c \otimes_{A^{op}} c' \mapsto c \Delta_{\mathcal{C}}(c') =: cc'^{(1)} \otimes_A c'^{(2)},$$

as in (2.2). Let P be an A^{op} -subring in \mathcal{C} , i.e. a subalgebra such that $t(a) \in P$, for all $a \in A$. Assume that $\Delta_{\mathcal{C}}(p) \in \mathcal{C} \otimes_A P$ for all $p \in P$ (e.g. P is a left subcomodule of \mathcal{C}). Denote by P^+ the intersection of P with the kernel of $\varepsilon_{\mathcal{C}}$. Then the right ideal (hence A - A sub-bimodule) $P^+ \mathcal{C}$, generated by P^+ , is also a coideal. Indeed, in a right bialgebroid $\ker \varepsilon_{\mathcal{C}}$ is a right ideal, hence for $p \in P^+$ and $c \in \mathcal{C}$ we have $\varepsilon_{\mathcal{C}}(pc) = \varepsilon_{\mathcal{C}}(s(\varepsilon_{\mathcal{C}}(p))c) = 0$. Furthermore, using the Takeuchi axiom for a right bialgebroid in the second equality, one computes

$$\begin{aligned} \Delta_{\mathcal{C}}(pc) &= p^{(1)} c^{(1)} \otimes_A (p^{(2)} - t(\varepsilon_{\mathcal{C}}(p^{(2)}))) c^{(2)} + p^{(1)} c^{(1)} \otimes_A t(\varepsilon_{\mathcal{C}}(p^{(2)})) c^{(2)} \\ &= p^{(1)} c^{(1)} \otimes_A (p^{(2)} - t(\varepsilon_{\mathcal{C}}(p^{(2)}))) c^{(2)} + p^{(1)} s(\varepsilon_{\mathcal{C}}(p^{(2)})) c^{(1)} \otimes_A c^{(2)} \\ (3.3) \quad &= p^{(1)} c^{(1)} \otimes_A (p^{(2)} - t(\varepsilon_{\mathcal{C}}(p^{(2)}))) c^{(2)} + pc^{(1)} \otimes_A c^{(2)}. \end{aligned}$$

The second term in (3.3) is a (possibly zero) element of $P^+ \mathcal{C} \otimes_A \mathcal{C}$. The map $c \mapsto c - t(\varepsilon_{\mathcal{C}}(c))$ splits the inclusion $\ker \varepsilon_{\mathcal{C}} \subseteq \mathcal{C}$, and its restriction splits the inclusion $P^+ \subseteq P$. Hence the first term in (3.3) belongs to $\mathcal{C} \otimes_A P^+ \mathcal{C}$. Thus $P^+ \mathcal{C}$ is a coideal. Denote by $\mathcal{Q} := \mathcal{C}/P^+ \mathcal{C}$ the quotient coring and right \mathcal{C} -module. Let $\pi : \mathcal{C} \rightarrow \mathcal{Q}$ denote the canonical epimorphism, which is a homomorphism of A -corings and of right \mathcal{C} -modules (so in particular of left A -modules). The map π induces a \mathcal{Q} -comodule structure on \mathcal{C} , with coaction $\varrho^{\mathcal{C}} := (\mathcal{C} \otimes_A \pi) \circ \Delta_{\mathcal{C}}$. Denote the algebra of \mathcal{Q} -coinvariants in \mathcal{C} by B . Note that, similarly to the computations in (3.3), for all $p \in P$ and $c \in \mathcal{C}$,

$$\begin{aligned} \varrho^{\mathcal{C}}(pc) &= p^{(1)} c^{(1)} \otimes \pi((p^{(2)} - t(\varepsilon_{\mathcal{C}}(p^{(2)}))) c^{(2)}) + p^{(1)} c^{(1)} \otimes \pi(t(\varepsilon_{\mathcal{C}}(p^{(2)})) c^{(2)}) \\ &= pc^{(1)} \otimes \pi(c^{(2)}). \end{aligned}$$

Hence we have a sequence of algebra inclusions $t(A) \subseteq P \subseteq B \subseteq \mathcal{C}$. We claim that $B \subseteq \mathcal{C}$ is a \mathcal{Q} -Galois extension, that is, the canonical map

$$(3.4) \quad \text{can} : \mathcal{C} \otimes_B \mathcal{C} \rightarrow \mathcal{C} \otimes_A \mathcal{Q}, \quad c \otimes_B c' \mapsto cc'^{(1)} \otimes_A \pi(c'^{(2)}),$$

is bijective. Consider the composite map

$$(3.5) \quad \mathcal{C} \otimes_A \mathcal{C} \xrightarrow{\theta^{-1}} \mathcal{C} \otimes_{A^{op}} \mathcal{C} \longrightarrow \mathcal{C} \otimes_B \mathcal{C},$$

where the rightmost arrow denotes the canonical epimorphism induced by the algebra inclusion $A^{op} \cong t(A) \subseteq B$. We show that the map (3.5) factorises through $\mathcal{C} \otimes_A \mathcal{Q}$. Introduce the index notation $\theta^{-1}(1_{\mathcal{C}} \otimes_A c) =: c_- \otimes_{A^{op}} c_+$, where implicit summation is understood. Since θ is a right \mathcal{C} -comodule map, so is θ^{-1} . That is, for all $c \in \mathcal{C}$,

$$c_- \otimes_{A^{op}} c_+^{(1)} \otimes_A c_+^{(2)} = c^{(1)}_- \otimes_{A^{op}} c^{(1)}_+ \otimes_A c^{(2)}.$$

This implies that, for all $p \in P$,

$$p_- \otimes_{A^{op}} p_+ = p_- \otimes_{A^{op}} p_+^{(2)} t(\varepsilon_{\mathcal{C}}(p_+^{(1)})) = p^{(1)}_- \otimes_{A^{op}} p^{(2)} t(\varepsilon_{\mathcal{C}}(p^{(1)}_+))$$

is an element of $\mathcal{C} \otimes_{A^{op}} P$, so a (possibly zero) element of $\mathcal{C} \otimes_{A^{op}} B$. Hence, for all $c, c' \in \mathcal{C}$ and $p \in P^+$, the left \mathcal{C} -module map (3.5) takes $c' \otimes_A pc$ to

$$c'(pc)_- \otimes_B (pc)_+ = c' c_- p_- \otimes_B p_+ c_+ = c' c_- p_- p_+ \otimes_B c_+ = c' c_- s(\varepsilon_{\mathcal{C}}(p)) \otimes_B c_+ = 0.$$

In the first equality we used the multiplicativity of the map $\theta^{-1}(1_{\mathcal{C}} \otimes_A \bullet)$, i.e. that for all $c, c' \in \mathcal{C}$, $(cc')_- \otimes_{A^{op}} (cc')_+ = c'_- c_- \otimes_{A^{op}} c_+ c'_+$. In the third equality we used that $c_- c_+ = s(\varepsilon_{\mathcal{C}}(c))$. The last equality follows since $p \in P^+ \subseteq \ker \varepsilon_{\mathcal{C}}$. Thus we proved the existence of a map

$$\text{can}^{-1} : \mathcal{C} \otimes_A \mathcal{Q} \rightarrow \mathcal{C} \otimes_B \mathcal{C}, \quad c \otimes_A \pi(c') \mapsto cc'_- \otimes_B c'_+.$$

Since it is defined in terms of θ^{-1} , it is straightforward to see that it is the inverse of the canonical map (3.4).

By Lemma 3.5, corresponding to the \mathcal{Q} -Galois extension $B \subseteq \mathcal{C}$, there is an A - B pre-torsor structure on \mathcal{C} , with pre-torsor map

$$\mathcal{C} \rightarrow \mathcal{C} \otimes_A \mathcal{C} \otimes_B \mathcal{C}, \quad c \mapsto c^{(1)} \otimes_A c^{(2)}_- \otimes_B c^{(2)}_+.$$

Example 3.10. Consider a right entwining structure (T, \mathcal{C}, ψ) over an algebra A , such that the right regular T -module extends to an entwined module. That is, T is a right \mathcal{C} -comodule, with coaction $\varrho^T : T \rightarrow T \otimes_A \mathcal{C}$, $t \mapsto t^{[0]} \otimes_A t^{[1]}$ such that, for all $t, t' \in T$,

$$(3.6) \quad (tt')^{[0]} \otimes_A (tt')^{[1]} = t^{[0]} \psi(t^{[1]} \otimes_A t').$$

Assume that the range of the unit map $\alpha : A \rightarrow T$ of the A -ring T lies within the coinvariant subalgebra $B := T^{co\mathcal{C}}$. Furthermore, assume that there exists a left A -module right \mathcal{C} -comodule map $j : \mathcal{C} \rightarrow T$, which possesses a convolution inverse, i.e. an A - A bimodule map $\tilde{j} : \mathcal{C} \rightarrow T$, such that

$$j(c^{(1)}) \tilde{j}(c^{(2)}) = \alpha(\varepsilon_{\mathcal{C}}(c)) = \tilde{j}(c^{(1)}) j(c^{(2)}), \quad \text{for all } c \in \mathcal{C},$$

where $\varepsilon_{\mathcal{C}}$ denotes the counit of \mathcal{C} , and $\Delta_{\mathcal{C}}(c) = c^{(1)} \otimes_A c^{(2)}$. In this situation T is said to be a \mathcal{C} -cleft extension of B , cf. [7, Proposition 6.4]. A \mathcal{C} -cleft extension is \mathcal{C} -Galois by [7, Corollary 5.3]. Hence by Lemma 3.5 there is a corresponding A - B pre-torsor structure on T . The pre-torsor map is

$$\tau : T \rightarrow T \otimes_A T \otimes_B T, \quad t \mapsto t^{[0]} \otimes_A \tilde{j}(t^{[1]}) \otimes_B j(t^{[2]}).$$

Note that τ is well defined since j and \tilde{j} are A - A bimodule maps and B is an A -ring. It is obviously B - A bilinear. Also,

$$(T \otimes_A \mu_T)(\tau(t)) = t^{[0]} \otimes_A \tilde{j}(t^{[1]}) j(t^{[2]}) = t^{[0]} \otimes_A \alpha(\varepsilon_{\mathcal{C}}(t^{[1]})) = t^{[0]} \alpha(\varepsilon_{\mathcal{C}}(t^{[1]})) \otimes_A 1_T = t \otimes_A 1_T,$$

hence axiom (b) in Definition 3.1 holds. Axiom (a) is proven as follows. Similarly to [1, Lemma 4.7 1], the identity

$$(3.7) \quad \tilde{j}(c) 1_T^{[0]} \otimes_A 1_T^{[1]} = \psi(c^{(1)} \otimes_A \tilde{j}(c^{(2)}))$$

can be proven, for all $c \in \mathcal{C}$. Hence, for all $t \in T$,

$$\varrho(t^{[0]} \tilde{j}(t^{[1]})) = t^{[0]} \psi(t^{[1]} \otimes_A \tilde{j}(t^{[2]})) = t^{[0]} \tilde{j}(t^{[1]}) 1_T^{[0]} \otimes_A 1_T^{[1]}.$$

The first equality follows by (3.6) and the second one follows by (3.7). That is, $t^{[0]} \tilde{j}(t^{[1]}) \in B$, for all $t \in T$. Hence

$$\begin{aligned} (\mu_T \otimes_B T)(\tau(t)) &= t^{[0]} \tilde{j}(t^{[1]}) \otimes_B j(t^{[2]}) = 1_T \otimes_B t^{[0]} \tilde{j}(t^{[1]}) j(t^{[2]}) \\ &= 1_T \otimes_B t^{[0]} \alpha(\varepsilon_{\mathcal{C}}(t^{[1]})) = 1_T \otimes_B t. \end{aligned}$$

Finally, axiom (c) follows by the right \mathcal{C} -colinearity of j as

$$\begin{aligned} (T \otimes T \otimes \tau)(\tau(t)) &= t^{[0]} \otimes_A \tilde{j}(t^{[1]}) \otimes_B j(t^{[2]})^{[0]} \otimes_A \tilde{j}(j(t^{[2]})^{[1]}) \otimes_B j(j(t^{[2]})^{[2]}) \\ &= t^{[0]} \otimes_A \tilde{j}(t^{[1]}) \otimes_B j(t^{[2]}) \otimes_A \tilde{j}(t^{[3]}) \otimes_B j(t^{[4]}) = (\tau \otimes T \otimes T)(\tau(t)). \end{aligned}$$

4. EQUIVALENCES INDUCED BY PRE-TORSORS

Faithfully flat Hopf-Galois extensions play a central role in the description of k -linear monoidal equivalences between comodule categories of flat Hopf algebras over a commutative ring k . Extending results on commutative Hopf algebras in [26] and [13], in [34] and [35] Ulbrich established an equivalence of fibre functors $U : {}^{\mathcal{C}}\mathcal{M} \rightarrow \mathcal{M}_k$ with faithfully flat \mathcal{C} -Galois extensions $U(\mathcal{C})$ of k , for any k -flat Hopf algebra \mathcal{C} . If k is a field, then a reconstruction theorem in [36] implies that for any fibre functor U there exists a (unique up to an isomorphism) k -Hopf algebra \mathcal{D} , such that U factorises through a k -linear monoidal equivalence functor ${}^{\mathcal{C}}\mathcal{M} \rightarrow {}^{\mathcal{D}}\mathcal{M}$ and the forgetful functor ${}^{\mathcal{D}}\mathcal{M} \rightarrow \mathcal{M}_k$. Thus there is an equivalence between k -linear monoidal equivalence functors ${}^{\mathcal{C}}\mathcal{M} \rightarrow {}^{\mathcal{D}}\mathcal{M}$ and \mathcal{C} -Galois extensions of k .

If k is any commutative ring, then no reconstruction theorem for arbitrary fibre functors is available. However, as Schauenburg pointed out in [27] and [31], for any faithfully flat \mathcal{C} -Galois extension T of k there exists a (unique up to an isomorphism) k -Hopf algebra \mathcal{D} , such that T is a \mathcal{D} - \mathcal{C} bi-Galois extension of k . This observation was used in [27] to prove that the fibre functor ${}^{\mathcal{C}}\mathcal{M} \rightarrow \mathcal{M}_k$, induced by T , factorises through a k -linear monoidal equivalence functor ${}^{\mathcal{C}}\mathcal{M} \rightarrow {}^{\mathcal{D}}\mathcal{M}$ and the forgetful functor ${}^{\mathcal{D}}\mathcal{M} \rightarrow \mathcal{M}_k$. Furthermore, every k -linear monoidal equivalence functor ${}^{\mathcal{C}}\mathcal{M} \rightarrow {}^{\mathcal{D}}\mathcal{M}$ was shown to be induced by a faithfully flat \mathcal{D} - \mathcal{C} bi-Galois extension of k .

The first steps to extend the above theory to non-commutative base algebras (replacing the commutative ring k above), were made by Schauenburg in [28]. Faithfully flat \mathcal{C} -Galois extensions of an arbitrary algebra B , for a k -flat Hopf algebra \mathcal{C} , were proven to induce k -linear monoidal equivalences between the categories of comodules of the k -Hopf algebra \mathcal{C} and a B -bialgebroid \mathcal{D} .

Our aim in the next two sections is to develop a more symmetric study of functors induced by faithfully flat (bi-)Galois extensions. That is, to replace both Hopf algebras \mathcal{C} and \mathcal{D} above with bialgebroids over arbitrary, different base algebras A and B . The problem is divided into two parts. While the bialgebroid structures of \mathcal{C} and \mathcal{D} are essential for having a monoidal structure on the categories of their comodules, one can study more general functors between categories of comodules of corings, induced by faithfully flat A - B pre-torsors. By Theorem 3.4, a faithfully flat A - B pre-torsor T is a bi-Galois object, for a B -coring \mathcal{D} and an A -coring \mathcal{C} . In this section it is shown that (under some additional assumptions (iii) in Remark 4.7 below) the cotensor product $T \square_{\mathcal{C}} \bullet$ defines a functor ${}^{\mathcal{C}}\mathcal{M} \rightarrow {}^{\mathcal{D}}\mathcal{M}$. As a main result we prove that this functor is an equivalence of categories. In forthcoming Section 5 our study will be specialised to functors induced by A - B torsors, whose monoidal properties will be studied.

In a similar way as in [9, Theorem 2.7], to a right \mathcal{C} -Galois extension $B \subseteq T$ one associates a right entwining structure over A , consisting of the A -ring T , the A -coring \mathcal{C} and the entwining map

$$\mathcal{C} \otimes_A T \rightarrow T \otimes_A \mathcal{C}, \quad c \otimes_A t \mapsto \text{can}_{\mathcal{C}}(\text{can}_{\mathcal{C}}^{-1}(1_T \otimes_A c) \cdot t),$$

such that the right regular T -module extends to a right entwined module. In a symmetric way to a left \mathcal{D} -Galois extension $A \subseteq T$ there corresponds a left entwining structure over B , consisting of the B -ring T , the B -coring \mathcal{D} and the entwining map

$$T \otimes_B \mathcal{D} \rightarrow \mathcal{D} \otimes_B T, \quad t \otimes_B d \mapsto {}_{\mathcal{D}}\text{can}(t {}_{\mathcal{D}}\text{can}^{-1}(d \otimes_B 1_T)),$$

such that the left regular T -module extends to a left entwined module.

By Theorem 3.4, to a faithfully flat A - B pre-torsor T one can associate a right entwining structure $(T, \mathcal{C}, \psi_{\mathcal{C}})$ over A and a left entwining structure $(T, \mathcal{D}, \psi_{\mathcal{D}})$ over B in the ways described above. The corings \mathcal{C} and \mathcal{D} were determined in Theorem 3.4 (ii) and (iii), respectively. The entwining maps have the following explicit forms.

$$(4.1) \quad \psi_{\mathcal{C}} : \mathcal{C} \otimes_A T \rightarrow T \otimes_A \mathcal{C}, \quad \sum_i t_i \otimes_B u_i \otimes_A v \mapsto \sum_i t_i \tau(u_i v),$$

$$(4.2) \quad \psi_{\mathcal{D}} : T \otimes_B \mathcal{D} \rightarrow \mathcal{D} \otimes_B T, \quad t \otimes_B \sum_j u_j \otimes_A v_j \mapsto \sum_j \tau(t u_j) v_j.$$

By an easy extension of [10, 32.8 (2)] to a non-commutative base algebra A , $\mathcal{C} \otimes_A T$ is a right entwined module for the right entwining structure $(T, \mathcal{C}, \psi_{\mathcal{C}})$ over the algebra A , with right T -action $\mathcal{C} \otimes_A \mu_T$ and right \mathcal{C} -coaction $(\mathcal{C} \otimes_A \psi_{\mathcal{C}}) \circ (\Delta_{\mathcal{C}} \otimes_A T)$. Symmetrically, $T \otimes_B \mathcal{D}$ is a left entwined module for the left entwining structure $(T, \mathcal{D}, \psi_{\mathcal{D}})$ over the algebra B , with left T -action $\mu_T \otimes_B \mathcal{D}$ and left \mathcal{D} -coaction $(\psi_{\mathcal{D}} \otimes_B \mathcal{D}) \circ (T \otimes_B \Delta_{\mathcal{D}})$.

Proposition 4.1. *Let T be a faithfully flat A - B pre-torsor and \mathcal{C} and \mathcal{D} the associated A - and B -corings, respectively. The following A - B sub-bimodules of $T \otimes_B T \otimes_A T$ coincide:*

- (i) *The coinvariants of the left entwined module $T \otimes_B \mathcal{D}$ for the left entwining structure $(T, \mathcal{D}, \psi_{\mathcal{D}})$ over the algebra B ;*
- (ii) *The coinvariants of the right entwined module $\mathcal{C} \otimes_A T$ for the right entwining structure $(T, \mathcal{C}, \psi_{\mathcal{C}})$ over the algebra A ;*
- (iii) *The intersection of $\mathcal{C} \otimes_A T$ and $T \otimes_B \mathcal{D}$.*

The A - B bimodule in (i)-(iii) will be denoted by \bar{T} in what follows.

Proof. We prove the equality of the bimodules (i) and (iii). The equality of (ii) and (iii) follows by a symmetrical reasoning. By the form (4.2) of the entwining map $\psi_{\mathcal{D}}$, the left \mathcal{D} -coaction $T \otimes_B \mathcal{D} \rightarrow \mathcal{D} \otimes_B T \otimes_B \mathcal{D}$ comes out explicitly as

$$\sum_j t_j \otimes_B u_j \otimes_A v_j \mapsto \sum_j (t_j u_j^{\langle 1 \rangle})^{\langle 1 \rangle} \otimes_A (t_j u_j^{\langle 1 \rangle})^{\langle 2 \rangle} \otimes_B (t_j u_j^{\langle 1 \rangle})^{\langle 3 \rangle} u_j^{\langle 2 \rangle} \otimes_B u_j^{\langle 3 \rangle} \otimes_A v_j.$$

If the element $\sum_j t_j \otimes_B u_j \otimes_A v_j \in T \otimes_B \mathcal{D}$ belongs to $\mathcal{C} \otimes_A T$, i.e. the kernel of the map $\omega \otimes_A T$ in Lemma 3.6, then

$$(4.3) \quad \begin{aligned} \sum_j (t_j u_j^{\langle 1 \rangle})^{\langle 1 \rangle} \otimes_A (t_j u_j^{\langle 1 \rangle})^{\langle 2 \rangle} \otimes_B (t_j u_j^{\langle 1 \rangle})^{\langle 3 \rangle} u_j^{\langle 2 \rangle} \otimes_B u_j^{\langle 3 \rangle} \otimes_A v_j \\ = \sum_j 1_T^{\langle 1 \rangle} \otimes 1_T^{\langle 2 \rangle} \otimes 1_T^{\langle 3 \rangle} t_j \otimes_B u_j \otimes_A v_j, \end{aligned}$$

hence $\sum_j t_j \otimes_B u_j \otimes_A v_j$ is a coinvariant in $T \otimes_B \mathcal{D}$. Conversely, assume that $\sum_j t_j \otimes_B u_j \otimes_A v_j$ is a coinvariant in $T \otimes_B \mathcal{D}$, i.e. satisfies (4.3). Applying $T \otimes_A \mu_T \otimes_B T \otimes_A T$ to both sides of (4.3) and using Definition 3.1 (b), we conclude that $\sum_j t_j \otimes_B u_j \otimes_A v_j$ lies in the kernel of $\omega \otimes_A T$, that is $\mathcal{C} \otimes_A T$. \square

Lemma 4.2. *Let T be a faithfully flat A - B pre-torsor with structure map τ , and \mathcal{C} and \mathcal{D} the associated A - and B -corings, respectively. The A - B bimodule \bar{T} in Proposition 4.1 is a \mathcal{C} - \mathcal{D} bicomodule. Both the left \mathcal{C} -coaction and the right \mathcal{D} -coaction are given by the restriction of $T \otimes_B \tau \otimes_A T$.*

Proof. Since $(\tau \otimes_A T)(\mathcal{D}) \subseteq \mathcal{D} \otimes_B \mathcal{D}$ and $\bar{T} \subseteq T \otimes_B \mathcal{D}$, it follows that $(T \otimes_B \tau \otimes_A T)(\bar{T}) \subseteq T \otimes_B \mathcal{D} \otimes_B \mathcal{D}$. Furthermore, by Definition 3.1 (c), for $\sum_j t_j \otimes_B u_j \otimes_A v_j \in \bar{T} \subseteq \mathcal{C} \otimes_A T$,

$$\begin{aligned} & \sum_j (t_j u_j^{\langle 1 \rangle \langle 1 \rangle})^{\langle 1 \rangle} \otimes_A (t_j u_j^{\langle 1 \rangle \langle 1 \rangle})^{\langle 2 \rangle} \otimes_B (t_j u_j^{\langle 1 \rangle \langle 1 \rangle})^{\langle 3 \rangle} u_j^{\langle 1 \rangle \langle 2 \rangle} \otimes_B u_j^{\langle 1 \rangle \langle 3 \rangle} \otimes_A u_j^{\langle 2 \rangle} \otimes_B u_j^{\langle 3 \rangle} \otimes_A v_j \\ &= \sum_j (t_j u_j^{\langle 1 \rangle})^{\langle 1 \rangle} \otimes_A (t_j u_j^{\langle 1 \rangle})^{\langle 2 \rangle} \otimes_B (t_j u_j^{\langle 1 \rangle})^{\langle 3 \rangle} u_j^{\langle 2 \rangle} \otimes_B u_j^{\langle 3 \rangle \langle 1 \rangle} \otimes_A u_j^{\langle 3 \rangle \langle 2 \rangle} \otimes_B u_j^{\langle 3 \rangle \langle 3 \rangle} \otimes_A v_j \\ &= \sum_j 1_T^{\langle 1 \rangle} \otimes_A 1_T^{\langle 2 \rangle} \otimes_B 1_T^{\langle 3 \rangle} t_j \otimes_B u_j^{\langle 1 \rangle} \otimes_A u_j^{\langle 2 \rangle} \otimes_B u_j^{\langle 3 \rangle} \otimes_A v_j. \end{aligned}$$

That is, $\sum_j t_j \otimes_B \tau(u_j) \otimes_A v_j$ is a coinvariant in the left entwined module $T \otimes_B \mathcal{D} \otimes_B \mathcal{D}$, for the left entwining structure $(T, \mathcal{D}, \psi_{\mathcal{D}})$. By a left handed version of Lemma 2.1, ${}^{co\mathcal{D}}(T \otimes_B \mathcal{D} \otimes_B \mathcal{D}) = {}^{co\mathcal{D}}(T \otimes_B \mathcal{D}) \otimes_B \mathcal{D}$, hence $(T \otimes_B \tau \otimes_A T)(\bar{T}) \subseteq \bar{T} \otimes_B \mathcal{D}$. The map $T \otimes_B \tau \otimes_A T$ is obviously right B -linear. Its counitality and coassociativity follow by Definition 3.1 (b) and (c), respectively. This proves that \bar{T} is a right \mathcal{D} -comodule, with coaction given by the restriction of $T \otimes_B \tau \otimes_A T$. By a symmetrical reasoning \bar{T} is also a left \mathcal{C} -comodule with coaction given by the restriction of $T \otimes_B \tau \otimes_A T$. The commutativity of the left \mathcal{C} -coaction and the right \mathcal{D} -coaction in \bar{T} follows by Definition 3.1 (c). \square

Corollary 4.3. *Let T be a faithfully flat A - B pre-torsor and \mathcal{C} and \mathcal{D} the associated A - and B -corings, respectively. Let T be the \mathcal{C} - \mathcal{D} bicomodule in Lemma 4.2.*

- (1) $T \otimes_B \mathcal{D}$ and $T \otimes_A \bar{T}$ are isomorphic as left T -modules left \mathcal{D} -comodules and right \mathcal{D} -comodules.
- (2) $\mathcal{C} \otimes_A T$ and $\bar{T} \otimes_B T$ are isomorphic as right T -modules right \mathcal{C} -comodules and left \mathcal{C} -comodules.

Proof. By the Galois Coring Structure Theorem [10, 28.19 (2) (a) \Rightarrow (c)], the ‘coinvariants’ functor ${}_T^P \mathcal{M}(\psi_{\mathcal{D}}) \rightarrow {}_A \mathcal{M}$ is an equivalence, with inverse $T \otimes_A \bullet$. Hence the counit of the adjunction is an isomorphism of left T -modules and left \mathcal{D} -comodules. In particular, by characterisation of \bar{T} in Proposition 4.1 (i),

$$(4.4) \quad T \otimes_A \bar{T} \rightarrow T \otimes_B \mathcal{D}, \quad t' \otimes \sum_j t_j \otimes_B u_j \otimes_A v_j \mapsto \sum_j t' t_j \otimes_B u_j \otimes_A v_j,$$

is an isomorphism of left T -modules and left \mathcal{D} -comodules. Its inverse has the form

$$(4.5) \quad T \otimes_B \mathcal{D} \rightarrow T \otimes_A \bar{T}, \quad t \otimes \sum_j u_j \otimes_A v_j \mapsto \sum_j t u_j^{\langle 1 \rangle} \otimes_A u_j^{\langle 2 \rangle} \otimes_B u_j^{\langle 3 \rangle} \otimes_A v_j.$$

Since (4.4) is obviously colinear with respect to the right \mathcal{D} -comodule structures of $T \otimes_A \bar{T}$ and $T \otimes_B \mathcal{D}$, defined via their second factors, this completes the proof of claim (1). Part (2) is proven symmetrically. \square

Theorem 4.4. *Let T be a faithfully flat A - B pre-torsor and \mathcal{C} and \mathcal{D} the associated A - and B -corings, respectively. Let \bar{T} be the \mathcal{C} - \mathcal{D} bicomodule in Lemma 4.2. Then*

- (1) $T \square_{\mathcal{C}} \bar{T}$ is a \mathcal{D} - \mathcal{D} bicomodule via the \mathcal{D} -coactions of T and \bar{T} , and $T \square_{\mathcal{C}} \bar{T} \cong \mathcal{D}$ as \mathcal{D} - \mathcal{D} bicomodules.
- (2) $\bar{T} \square_{\mathcal{D}} T$ is a \mathcal{C} - \mathcal{C} bicomodule via the \mathcal{C} -coactions of \bar{T} and T , and $\bar{T} \square_{\mathcal{D}} T \cong \mathcal{C}$ as \mathcal{C} - \mathcal{C} bicomodules.

Proof. Consider a map

$$(4.6) \quad \varpi : T \square_{\mathcal{C}} \bar{T} \rightarrow \mathcal{D}, \quad \sum_j t_j \otimes_A u_j \otimes_B v_j \otimes_A w_j \mapsto \sum_j t_j u_j v_j \otimes_A w_j = \sum_j t_j \otimes_A u_j v_j w_j,$$

where the last equality follows by $\sum_j t_j \otimes_A u_j \otimes_B v_j \otimes_A w_j \in T \square_{\mathcal{C}} \bar{T} \subseteq T \otimes_A \mathcal{C} \otimes_A T$, hence $\sum_j t_j \otimes_A u_j v_j \otimes_A w_j \in T \otimes_A A \otimes_A T$. By the defining equaliser property of the cotensor product $T \square_{\mathcal{C}} \bar{T}$, Definition 3.1 (a), and since $\bar{T} \subseteq T \otimes_B \mathcal{D}$,

$$\begin{aligned} \sum_j t_j^{\langle 1 \rangle} \otimes_A t_j^{\langle 2 \rangle} \otimes_B t_j^{\langle 3 \rangle} u_j v_j w_j &= \sum_j t_j \otimes_A u_j \otimes_B v_j^{\langle 1 \rangle} v_j^{\langle 2 \rangle} v_j^{\langle 3 \rangle} w_j \\ &= \sum_j t_j \otimes_A u_j \otimes_B v_j w_j = \sum_j t_j \otimes_A u_j v_j w_j \otimes_B 1_T. \end{aligned}$$

Hence the range of the map (4.6) is in \mathcal{D} , indeed. Since $\tau(T)$ lies within $T \otimes_A \mathcal{C}$, it follows that $(\tau \otimes_A T)(\mathcal{D})$ lies within $T \otimes_A \mathcal{C} \otimes_A T$. On the other hand, $(\tau \otimes_A T)(\mathcal{D})$ lies within $\mathcal{D} \otimes_B \mathcal{D} \subseteq T \otimes_A T \otimes_B \mathcal{D}$, hence characterisation of \bar{T} in Proposition 4.1 (iii), together with the flatness of T as a right A -module, implies that $(\tau \otimes_A T)(\mathcal{D})$ is a subset of $T \otimes_A \bar{T}$. Furthermore, by Definition 3.1 (c), the restriction of $\tau \otimes_A T$ defines a map $\mathcal{D} \rightarrow T \square_{\mathcal{C}} \bar{T}$. We claim that this map is the inverse of ϖ in (4.6). By Definition 3.1 (b) (or (a)), the restriction of $\varpi \circ (\tau \otimes_A T)$ to \mathcal{D} is equal to the identity map in \mathcal{D} . On the other hand, for $x = \sum_j t_j \otimes_A u_j \otimes_B v_j \otimes_A w_j \in T \square_{\mathcal{C}} \bar{T}$,

$$(\tau \otimes_A T)(\varpi(x)) = \sum_j t_j^{\langle 1 \rangle} \otimes_A t_j^{\langle 2 \rangle} \otimes_B t_j^{\langle 3 \rangle} u_j v_j \otimes_A w_j = \sum_j t_j \otimes_A u_j \otimes_B v_j^{\langle 1 \rangle} v_j^{\langle 2 \rangle} v_j^{\langle 3 \rangle} \otimes_A w_j = x.$$

Here the second equality follows by the definition of the cotensor product $T \square_{\mathcal{C}} \bar{T}$ and the third one follows by Definition 3.1 (b). This proves that the map (4.6) is bijective. Next take any $x = \sum_j t_j \otimes_A u_j \otimes_B v_j \otimes_A w_j \in T \square_{\mathcal{C}} \bar{T}$ and compute

$$\begin{aligned} (\mathcal{D} \otimes_B \varpi^{-1})(\Delta_{\mathcal{D}}(\varpi(x))) &= \sum_j t_j^{\langle 1 \rangle} \otimes_A t_j^{\langle 2 \rangle} \otimes_B t_j^{\langle 3 \rangle \langle 1 \rangle} \otimes_A t_j^{\langle 3 \rangle \langle 2 \rangle} \otimes_B t_j^{\langle 3 \rangle \langle 3 \rangle} u_j v_j \otimes_A w_j \\ &= \sum_j t_j^{\langle 1 \rangle \langle 1 \rangle} \otimes_A t_j^{\langle 1 \rangle \langle 2 \rangle} \otimes_B t_j^{\langle 1 \rangle \langle 3 \rangle} \otimes_A t_j^{\langle 2 \rangle} \otimes_B t_j^{\langle 3 \rangle} u_j v_j \otimes_A w_j \\ &= \sum_j t_j^{\langle 1 \rangle} \otimes_A t_j^{\langle 2 \rangle} \otimes_B t_j^{\langle 3 \rangle} \otimes_A u_j \otimes_B v_j^{\langle 1 \rangle} v_j^{\langle 2 \rangle} v_j^{\langle 3 \rangle} \otimes_A w_j \\ &= \sum_j t_j^{\langle 1 \rangle} \otimes_A t_j^{\langle 2 \rangle} \otimes_B t_j^{\langle 3 \rangle} \otimes_A u_j \otimes_B v_j \otimes_A w_j = (\tau \otimes_A \bar{T})(x). \end{aligned}$$

The second equality follows by Definition 3.1 (c), the third one follows by the equaliser property of the cotensor product $T \square_{\mathcal{C}} \bar{T}$ and the fourth one does by Definition 3.1 (b). The above computation confirms that $T \square_{\mathcal{C}} \bar{T}$ is a left \mathcal{D} -comodule with the coaction given by the restriction of $\tau \otimes_A \bar{T}$ and that ϖ is left \mathcal{D} -colinear. The fact that $T \square_{\mathcal{C}} \bar{T}$

is a right \mathcal{D} -comodule and the right \mathcal{D} -colinearity of ϖ are proven by a similar computation, for $x = \sum_j t_j \otimes_A u_j \otimes_B v_j \otimes_A w_j \in T \square_{\mathcal{C}} \bar{T}$.

$$\begin{aligned}
(\varpi^{-1} \otimes_B \mathcal{D})(\Delta_{\mathcal{D}}(\varpi(x))) &= \sum_j t_j^{\langle 1 \rangle \langle 1 \rangle} \otimes_A t_j^{\langle 1 \rangle \langle 2 \rangle} \otimes_B t_j^{\langle 1 \rangle \langle 3 \rangle} \otimes_A t_j^{\langle 2 \rangle} \otimes_B t_j^{\langle 3 \rangle} u_j v_j \otimes_A w_j \\
&= \sum_j t_j^{\langle 1 \rangle} \otimes_A t_j^{\langle 2 \rangle} \otimes_B t_j^{\langle 3 \rangle \langle 1 \rangle} \otimes_A t_j^{\langle 3 \rangle \langle 2 \rangle} \otimes_B t_j^{\langle 3 \rangle \langle 3 \rangle} u_j v_j \otimes_A w_j \\
&= \sum_j t_j \otimes_A u_j \otimes_B v_j^{\langle 1 \rangle \langle 1 \rangle} \otimes_A v_j^{\langle 1 \rangle \langle 2 \rangle} \otimes_B v_j^{\langle 1 \rangle \langle 3 \rangle} v_j^{\langle 2 \rangle} v_j^{\langle 3 \rangle} \otimes_A w_j \\
&= \sum_j t_j \otimes_A u_j \otimes_B v_j^{\langle 1 \rangle} \otimes_A v_j^{\langle 2 \rangle} \otimes_B v_j^{\langle 3 \rangle} \otimes_A w_j \\
&= (T \otimes_A (T \otimes_B \tau \otimes_A T))(x).
\end{aligned}$$

Here again, the second equality follows by Definition 3.1 (c), the third one follows by the equaliser property of the cotensor product $T \square_{\mathcal{C}} \bar{T}$ and the fourth one does by Definition 3.1 (b) and the right A -linearity of τ . This completes the proof of part (1). Part (2) follows by symmetrical arguments. \square

Lemma 4.5. *Let T be a faithfully flat A - B pre-torsor.*

- (1) *If T is a flat right B -module, then the A -coring \mathcal{C} , associated to T in Theorem 3.4 (ii), is a flat right A -module.*
- (2) *If T is a flat left A -module, then the B -coring \mathcal{D} , associated to T in Theorem 3.4 (iii), is a flat left B -module.*

Proof. Recall that for any ring extension $A \subseteq T$, the corresponding restriction of scalars (forgetful) functor ${}_T\mathcal{M} \rightarrow {}_A\mathcal{M}$ is faithful, hence it reflects monomorphisms. On the other hand, it possesses a left adjoint (the induction functor $T \otimes_A \bullet$), hence it preserves monomorphisms as well.

T is a faithfully flat right A -module by assumption. Hence \mathcal{C} is a flat right A -module, i.e. the functor $\mathcal{C} \otimes_A \bullet : {}_A\mathcal{M} \rightarrow \mathcal{M}_k$ preserves monomorphisms, if and only if the functor $T \otimes_A \mathcal{C} \otimes_A \bullet : {}_A\mathcal{M} \rightarrow \mathcal{M}_k$ preserves monomorphisms. By the right \mathcal{C} -Galois property of the extension $B \subseteq T$ (cf. Theorem 3.4), the right A -modules $T \otimes_A \mathcal{C}$ and $T \otimes_B T$ are isomorphic. Hence the functors $T \otimes_A \mathcal{C} \otimes_A \bullet : {}_A\mathcal{M} \rightarrow \mathcal{M}_k$ and $T \otimes_B T \otimes_A \bullet : {}_A\mathcal{M} \rightarrow \mathcal{M}_k$ are naturally isomorphic. The flatness of \mathcal{C} as a right A -module follows by the assumption that both functors $T \otimes_A \bullet : {}_A\mathcal{M} \rightarrow {}_B\mathcal{M}$ and $T \otimes_B \bullet : {}_B\mathcal{M} \rightarrow \mathcal{M}_k$ preserve monomorphisms, hence so does their composite. This completes the proof of claim (1). Assertion (2) follows by a symmetrical reasoning. \square

In light of Corollary 4.3, analogous considerations to those used to prove Lemma 4.5, lead to the following

Lemma 4.6. *Let T be a faithfully flat A - B pre-torsor. Assume that T is faithfully flat also as a right B -module. Then the associated B -coring \mathcal{D} is a flat right B -module if and only if the right B -module \bar{T} in Proposition 4.1 is flat.*

Remark 4.7. Consider an A - B pre-torsor T with structure map τ . If both A and B coincide with the ground ring k , then the following properties of T are equivalent by Lemmas 4.5 and 4.6.

- (i) T is a faithfully flat pre-torsor, i.e. a faithfully flat left B -module and right A -module.
- (ii) T is a faithfully flat left and right B -module and right A -module.
- (iii) T is a faithfully flat left and right B -module and right A -module, and the left B -module map $\tau \otimes_A M - T \otimes_A {}^M\varrho$ is $\mathcal{D} \otimes_B \mathcal{D}$ -pure, for any left \mathcal{C} -comodule M , with coaction ${}^M\varrho$.
- (iv) T is a faithfully flat left and right B -module and right A -module, and the associated B -coring in Theorem 3.4 is a flat right B -module.
- (v) T is a faithfully flat left and right B -module and right A -module, and the associated A - B bimodule \bar{T} in Proposition 4.1 is a flat right B -module.

For arbitrary base algebras A and B , properties (i)-(v) seem no longer equivalent. Only implications (i) \Leftarrow (ii) \Leftarrow (iii) \Leftarrow (iv) \Leftrightarrow (v) are easily proven. Indeed, (iv) \Leftrightarrow (v) is proven in Lemma 4.6. If \mathcal{D} is a flat right B -module then so is $\mathcal{D} \otimes_B \mathcal{D}$. Hence any left B -module map is $\mathcal{D} \otimes_B \mathcal{D}$ -pure. This proves (iv) \Rightarrow (iii). The remaining implications (iii) \Rightarrow (ii) \Rightarrow (i) are obvious. Some theorems can be proven by assuming a weaker one of the above properties, in other cases a stronger one is needed.

Corollary 4.8. *Let T be an A - B pre-torsor obeying properties (iii) in Remark 4.7. Then the categories of left comodules of the associated A - and B -corings \mathcal{C} and \mathcal{D} are equivalent. The inverse equivalences between them are given by the cotensor products $T \square_{\mathcal{C}} \bullet : {}^{\mathcal{C}}\mathcal{M} \rightarrow {}^{\mathcal{D}}\mathcal{M}$ and $\bar{T} \square_{\mathcal{D}} \bullet : {}^{\mathcal{D}}\mathcal{M} \rightarrow {}^{\mathcal{C}}\mathcal{M}$, respectively, where \bar{T} is the \mathcal{C} - \mathcal{D} bicomodule in Lemma 4.2.*

Proof. By the $\mathcal{D} \otimes_B \mathcal{D}$ -purity of the equalisers defining \mathcal{C} -cotensor products with T , there is a functor $T \square_{\mathcal{C}} \bullet : {}^{\mathcal{C}}\mathcal{M} \rightarrow {}^{\mathcal{D}}\mathcal{M}$. By Lemma 4.5 (1), \mathcal{C} is a flat right A -module, so the \mathcal{C} - \mathcal{D} bicomodule \bar{T} in Lemma 4.2 induces another functor $\bar{T} \square_{\mathcal{D}} \bullet : {}^{\mathcal{D}}\mathcal{M} \rightarrow {}^{\mathcal{C}}\mathcal{M}$. By a reasoning similar to [27, Section 2], the following sequence of isomorphisms holds, for any left \mathcal{D} -comodule M ,

$$\begin{aligned} T \otimes_B [(T \square_c \bar{T}) \square_{\mathcal{D}} M] &\cong [T \otimes_B (T \square_c \bar{T})] \square_{\mathcal{D}} M \cong [(T \otimes_B T) \square_c \bar{T}] \square_{\mathcal{D}} M \cong [(T \otimes_A \mathcal{C}) \square_c \bar{T}] \square_{\mathcal{D}} M \\ &\cong (T \otimes_A \bar{T}) \square_{\mathcal{D}} M \cong T \otimes_A (\bar{T} \square_{\mathcal{D}} M) \cong (T \otimes_A \mathcal{C}) \square_c (\bar{T} \square_{\mathcal{D}} M) \\ &\cong (T \otimes_B T) \square_c (\bar{T} \square_{\mathcal{D}} M) \cong T \otimes_B [T \square_c (\bar{T} \square_{\mathcal{D}} M)]. \end{aligned}$$

The first two and the last isomorphisms follow by the flatness of T as a right B -module. The fifth one follows by the flatness of T as a right A -module. The third and the penultimate equivalences follow by the right \mathcal{C} -Galois property of the extension $B \subseteq T$ (cf. Theorem 3.4). Since T is a faithfully flat right B -module by assumption, this implies that $(T \square_c \bar{T}) \square_{\mathcal{D}} M \cong T \square_c (\bar{T} \square_{\mathcal{D}} M)$, for any left \mathcal{D} -comodule M . Together with Theorem 4.4 (1), this proves that the composite of the functors $\bar{T} \square_{\mathcal{D}} \bullet$ and $T \square_{\mathcal{C}} \bullet$ is naturally isomorphic to the identity functor ${}^{\mathcal{D}}\mathcal{M}$. By Corollary 4.3 (1), an analogous argument proves that $T \otimes_A [(\bar{T} \square_{\mathcal{D}} T) \square_{\mathcal{C}} N] \cong T \otimes_A [\bar{T} \square_{\mathcal{D}} (T \square_{\mathcal{C}} N)]$, for any left \mathcal{C} -comodule N . Since T is a faithfully flat right A -module, we conclude that $(\bar{T} \square_{\mathcal{D}} T) \square_{\mathcal{C}} N \cong \bar{T} \square_{\mathcal{D}} (T \square_{\mathcal{C}} N)$. Hence it follows by Theorem 4.4 (2) that the composite of the functors $T \square_{\mathcal{C}} \bullet$ and $\bar{T} \square_{\mathcal{D}} \bullet$ is naturally isomorphic to the identity functor on ${}^{\mathcal{C}}\mathcal{M}$. \square

In the rest of the section we study the particular case when the entwining maps $\psi_{\mathcal{C}}$ in (4.1) and $\psi_{\mathcal{D}}$ in (4.2) are bijective. By the standard entwining structure arguments,

$(T, \mathcal{C}, \psi_{\mathcal{C}}^{-1})$ is a left entwining structure over A , and T is a left entwined module with the left regular T -action and the left \mathcal{C} -coaction

$$(4.7) \quad {}^T\varrho : T \rightarrow \mathcal{C} \otimes_A T, \quad t \mapsto \psi_{\mathcal{C}}^{-1}(t\tau(1_T)).$$

The coinvariants of T as a left \mathcal{C} -comodule coincide with the coinvariants B of T as a right \mathcal{C} -comodule. The left \mathcal{C} -Galois property of the algebra extension $B \subseteq T$ is equivalent to its right \mathcal{C} -Galois property. The canonical maps are related by the entwining map, i.e.

$$(4.8) \quad \text{can}_{\mathcal{C}} = \psi_{\mathcal{C}} \circ {}_c\text{can}.$$

By these considerations, if for a faithfully flat A - B pre-torsor T , the entwining maps (4.1) and (4.2) are bijective, then $B \subseteq T$ is a left \mathcal{C} -Galois extension and $A \subseteq T$ is a right \mathcal{D} -Galois extension. In fact one can prove more.

Theorem 4.9. *Let T be a faithfully flat A - B pre-torsor and \mathcal{C} and \mathcal{D} the associated corings in Theorem 3.4. Assume that the entwining maps (4.1) and (4.2) are bijective. Then the \mathcal{C} - \mathcal{D} bicomodule \bar{T} in Lemma 4.2 is isomorphic to T . Therefore T is a \mathcal{C} - \mathcal{D} bi-Galois object, hence in particular a B - A pre-torsor.*

Proof. For any element $t \otimes_B u \otimes_A v$ of $T \otimes_B T \otimes_A T$,

$$\begin{aligned} (T \otimes_A \psi_{\mathcal{C}})((\text{can}_{\mathcal{C}} \otimes T)(t \otimes_B u \otimes_A v)) &= tu^{(1)} \otimes_A u^{(2)} \tau(u^{(3)}v) = (T \otimes_A \text{can}_{\mathcal{C}})(t\tau(u)v), \\ (\psi_{\mathcal{D}} \otimes_B T)((T \otimes_B \text{can})(t \otimes_B u \otimes_A v)) &= \tau(tu^{(1)})u^{(2)} \otimes_B u^{(3)}v = (\text{can}_{\mathcal{D}} \otimes_B T)(t\tau(u)v). \end{aligned}$$

Therefore, by (4.8) and its analogue for the \mathcal{D} -Galois extension $A \subseteq T$,

$$(4.9) \quad (T \otimes_A {}_c\text{can}^{-1}) \circ (\text{can}_{\mathcal{C}} \otimes T) = (\text{can}_{\mathcal{D}}^{-1} \otimes_B T) \circ (T \otimes_B \text{can}).$$

Taking the inverses of both sides of (4.9) and evaluating on an element $1_T \otimes_A t \otimes_B 1_T$ of $T \otimes_A T \otimes_B T$, we conclude that the left \mathcal{C} -coaction and the right \mathcal{D} -coaction in T map an element $t \in T$ to the same element of $T \otimes_B T \otimes_A T$. For these equal coactions we use the notation $\bar{\tau} : T \rightarrow T \otimes_B T \otimes_A T$. By the counitality of the left \mathcal{C} -coaction or the right \mathcal{D} -coaction in T , $\bar{\tau}$ is an injective A - B bimodule map $T \rightarrow T \otimes_B T \otimes_A T$. As its range lies both in $\mathcal{C} \otimes_A T$ and $T \otimes_B \mathcal{D}$, $\bar{\tau}(T) \subseteq \bar{T}$. In order to prove the converse inclusion, take $\sum_i u_i \otimes_B v_i \otimes_A t_i \in \bar{T}$ and compute

$$\begin{aligned} \bar{\tau}(\sum_i u_i v_i t_i) &= \psi_{\mathcal{C}}^{-1}(\sum_i u_i v_i t_i \tau(1_T)) = \psi_{\mathcal{C}}^{-1}(\sum_i u_i v_i^{(1)} v_i^{(2)} \tau(v_i^{(3)} t_i)) \\ &= \psi_{\mathcal{C}}^{-1}(\sum_i u_i \tau(v_i t_i)) = \sum_i u_i \otimes_B v_i \otimes_A t_i. \end{aligned}$$

The second equality follows by the property that $\sum_i u_i \otimes_B v_i \otimes_A t_i$ is an element of $\bar{T} \subseteq T \otimes_B \mathcal{D}$, the third one does by Definition 3.1 (a) and the left B -linearity of τ , and the last one follows by the form (4.1) of $\psi_{\mathcal{C}}$. Thus $\bar{\tau}$ is a bijection $T \rightarrow \bar{T}$. Its left \mathcal{C} -colinearity and right \mathcal{D} -colinearity follow by the coassociativity of the left \mathcal{C} -coaction $\bar{\tau}$ and the right \mathcal{D} -coaction $\bar{\tau}$ in \bar{T} , respectively. \square

Note that it follows by Theorem 4.9 that if for a faithfully flat A - B pre-torsor T the maps (4.1) and (4.2) are bijective, then properties (ii), (iii), (iv) and (v) in Remark 4.7 are all equivalent to each other.

The content of the following lemma was observed in [4, Remark 2.4 (1)].

Lemma 4.10. *Let T be a faithfully flat A - B pre-torsor.*

- (1) If T is a faithfully flat right B -module and the entwining map (4.1) is bijective, then also the entwining map (4.2) is bijective.
- (2) If T is a faithfully flat left A -module and the entwining map (4.2) is bijective, then also the entwining map (4.1) is bijective.

Proof. In the case when the entwining map (4.1) is bijective, $T \otimes_A T$ is a left entwined module for the left entwining structure $(T, \mathcal{C}, \psi_{\mathcal{C}}^{-1})$ over A , via the left regular T -module structure of the first factor and the left \mathcal{C} -coaction $(\psi_{\mathcal{C}}^{-1} \otimes_A T) \circ (T \otimes_A {}^T\varrho)$, given in terms of the left \mathcal{C} -coaction (4.7) in T . Its coinvariants are the elements $\sum_j u_j \otimes_A v_j$ of $T \otimes_A T$, for which

$$(4.10) \quad \sum_j (\psi_{\mathcal{C}}^{-1} \otimes_A T)((T \otimes_A {}^T\varrho)(u_j \otimes_A v_j)) = \sum_j \psi_{\mathcal{C}}^{-1}(\tau(1_T))u_j \otimes_A v_j.$$

Applying $\psi_{\mathcal{C}} \otimes_A T$ to both sides of (4.10), we obtain the equivalent condition

$$(4.11) \quad (T \otimes_A {}^T\varrho)(\sum_j u_j \otimes_A v_j) = (\tau \otimes_A T)(\sum_j u_j \otimes_A v_j).$$

By Theorem 3.4, $B \subseteq T$ is a right \mathcal{C} -Galois extension. Hence one can apply the isomorphism $T \otimes_A \text{can}_{\mathcal{C}}^{-1} \circ \psi_{\mathcal{C}} = T \otimes_A T \otimes_B \mu_T$ to (4.11) to obtain

$$(T \otimes_A T \otimes_B \beta)(\sum_j u_j \otimes_A v_j) = (T \otimes_A T \otimes_B \mu_T)((\tau \otimes_A T)(\sum_j u_j \otimes_A v_j)).$$

This shows that the coinvariants of the left entwined module $T \otimes_A T$ coincide with the elements of \mathcal{D} . Since the T -coring $T \otimes_A \mathcal{C}$, associated to the right entwining structure $(T, \mathcal{C}, \psi_{\mathcal{C}})$, is a Galois coring, so is the isomorphic coring $\mathcal{C} \otimes_A T$, associated to the left entwining structure $(T, \mathcal{C}, \psi_{\mathcal{C}}^{-1})$. By the assumption that T is a faithfully flat right B -module, the Galois Coring Structure Theorem [10, 28.19 (2) (a) \Rightarrow (c)] implies the bijectivity of the map

$$(4.12) \quad T \otimes_B \mathcal{D} \rightarrow T \otimes_A T, \quad t \otimes \sum_j u_j \otimes_A v_j \mapsto \sum_j tu_j \otimes_A v_j,$$

which is simply the counit of the adjunction of the ‘coinvariants’ functor ${}^{\mathcal{C}}_T \mathcal{M} \rightarrow {}_B \mathcal{M}$ and the induction functor $T \otimes_B \bullet : {}_B \mathcal{M} \rightarrow {}^{\mathcal{C}}_T \mathcal{M}$, evaluated at the left entwined module $T \otimes_A T$. The map (4.12) is the inverse of the right \mathcal{D} -canonical map $\text{can}_{\mathcal{D}}$. Hence it is related via the \mathcal{D} -analogue of (4.8) to the left \mathcal{D} -canonical map ${}_{\mathcal{D}} \text{can} : T \otimes_A T \rightarrow \mathcal{D} \otimes_B T$, which is bijective by Theorem 3.4. Hence (4.2) is bijective, as stated in (1). Part (2) is proven by a symmetric argument. \square

Corollary 4.11. *Let T be an A - B pre-torsor, obeying properties (ii) in Remark 4.7. Assume that the entwining map (4.1) is bijective. Then T possesses a \mathcal{D} - \mathcal{C} bicomodule structure (cf. Theorem 3.4) and a \mathcal{C} - \mathcal{D} bicomodule structure (cf. Theorem 4.9). Furthermore, the functors $T \square_{\mathcal{D}} \bullet : {}^{\mathcal{D}} \mathcal{M} \rightarrow {}^{\mathcal{C}} \mathcal{M}$ and $T \square_{\mathcal{C}} \bullet : {}^{\mathcal{C}} \mathcal{M} \rightarrow {}^{\mathcal{D}} \mathcal{M}$ are inverse equivalences.*

5. A - B TORSORS AS MONOIDAL FUNCTORS

In this section we focus our attention on faithfully flat A - B torsors in the sense of [17, Definition 5.2.1]. Following [17, Theorem 5.2.10], in Theorem 5.2 we establish a bijective correspondence between faithfully flat A - B torsors and bi-Galois objects by

bialgebroids (actually \times_A - and \times_B -Hopf algebras), \mathcal{C} and \mathcal{D} . A characteristic feature of bialgebroids is the monoidality of the category of their comodules. In Theorems 5.4 and 5.6 we show that an A - B torsor T , with properties in Remark 4.7 (ii), induces a monoidal functor from ${}^{\mathcal{C}}\mathcal{M}$, the category of comodules of the associated \times_A -Hopf algebra \mathcal{C} , to the category of B - B bimodules. If T obeys properties (iii) in Remark 4.7, then the induced functor factorises through the category of \mathcal{D} -comodules, monoidally. What is more, by virtue of Corollary 4.8, it results in a monoidal equivalence between the comodule categories ${}^{\mathcal{C}}\mathcal{M}$ and ${}^{\mathcal{D}}\mathcal{M}$ (cf. Theorem 5.7). In contrast to the case when \mathcal{C} and \mathcal{D} are (flat) Hopf algebras, it is not known in general if all monoidal equivalence functors ${}^{\mathcal{C}}\mathcal{M} \rightarrow {}^{\mathcal{D}}\mathcal{M}$ arise in this way.

We start by recalling the following [17, Definition 5.2.1].

Definition 5.1. An A - B pre-torsor T with unit maps $\alpha : A \rightarrow T$ and $\beta : B \rightarrow T$ is called an A - B torsor if $\alpha(A)$ and $\beta(B)$ are commuting subalgebras in T and the structure map $\tau : T \rightarrow T \otimes_A T \otimes_B T$, $t \mapsto t^{(1)} \otimes_A t^{(2)} \otimes_B t^{(3)}$ obeys the following properties.

- (a) $\alpha(a)t^{(1)} \otimes_A t^{(2)} \otimes_B t^{(3)} = t^{(1)} \otimes_A t^{(2)}\alpha(a) \otimes_B t^{(3)}$;
- (b) $t^{(1)} \otimes_A \beta(b)t^{(2)} \otimes_B t^{(3)} = t^{(1)} \otimes_A t^{(2)} \otimes_B t^{(3)}\beta(b)$;
- (c) $\tau(tt') = t^{(1)}t'^{(1)} \otimes_A t'^{(2)}t^{(2)} \otimes_B t^{(3)}t'^{(3)}$;
- (d) $\tau(1_T) = 1_T \otimes_A 1_T \otimes_B 1_T$,

for all elements t and t' in T , a in A and b in B .

An A - B torsor is *faithfully flat* if it is faithfully flat as a right A -module and left B -module.

Note that axioms (a) and (b) are meaningful by the assumption that $\alpha(A)$ and $\beta(B)$ are commuting subalgebras in T . Axiom (c) makes sense in view of axioms (a) and (b). In order to simplify notation, we will not write out the unit maps α and β explicitly in the sequel.

The notion of an A - B torsor is made interesting by its relation to Galois extensions by bialgebroids. It was observed in [17, Theorem 5.2.10] that an A - B torsor T determines a left, and a right Galois extension, by two canonically associated bialgebroids, provided T is faithfully flat as a left and right module for both base algebras A and B (cf. Remark 3.3). In contrast, in Theorem 5.2 below we assume faithful flatness of T as a right A -module and a left B -module only. We also prove the converse of [17, Theorem 5.2.10], i.e. that a faithfully flat (left or right) Galois extension by a bialgebroid determines a torsor. Putting these results together, we prove that the notions of a faithfully flat A - B torsor, and that of a faithfully flat bi-Galois extension by bialgebroids, are equivalent. For the convenience of the reader we include (in a sketchy form) the complete proof, also of the parts which were obtained already in [17]. Instead of following the original arguments there (operating with the faithful flatness over its base algebra of a bialgebroid associated to a torsor), we make use of Theorem 3.4.

Theorem 5.2. *There is a bijective correspondence between the following sets of data:*

- (i) *faithfully flat A - B torsors T ;*
- (ii) *right \times_A -Hopf algebras \mathcal{C} and left faithfully flat right \mathcal{C} -Galois extensions $B \subseteq T$, such that T is a right faithfully flat A -ring;*

- (iii) left \times_B -Hopf algebras \mathcal{D} and right faithfully flat left \mathcal{D} -Galois extensions $A \subseteq T$, such that T is a left faithfully flat B -ring.

Proof. In light of Theorem 3.4, we need to show that a faithfully flat A - B pre-torsor T is a torsor if and only if the associated A -coring \mathcal{C} is a right \times_A -Hopf algebra and T is its right comodule algebra. Equivalently, if and only if the associated B -coring \mathcal{D} is a left \times_B -Hopf algebra and T is its left comodule algebra.

Assume first that T is a faithfully flat A - B torsor, with structure map τ . Use Definition 3.1 (a), Definition 5.1 (b) and the definition of \mathcal{C} , as the kernel of the map ω in Lemma 3.6, to see that, for $b \in B$ and $\sum_i u_i \otimes_B v_i \in \mathcal{C}$,

$$\sum_i u_i \otimes_B v_i b = \sum_i u_i v_i^{(1)} v_i^{(2)} \otimes_B v_i^{(3)} b = \sum_i u_i v_i^{(1)} b v_i^{(2)} \otimes_B v_i^{(3)} = \sum_i b u_i \otimes_B v_i.$$

This implies that, for any elements $\sum_i u_i \otimes_B v_i$ and $\sum_j u'_j \otimes_B v'_j$ in \mathcal{C} , $\sum_{i,j} u'_j u_i \otimes_B v_i v'_j$ is a well defined element of $T \otimes_B T$. Furthermore, it follows by Definition 5.1 (c) that it belongs to $\ker \omega = \mathcal{C}$. Since by Definition 5.1 (d) also $1_T \otimes_B 1_T$ is an element of \mathcal{C} , we conclude that \mathcal{C} is an algebra, with multiplication and unit inherited from the algebra $T^{op} \otimes_k T$. The right \mathcal{C} -coaction τ is multiplicative by Definition 5.1 (c) and unital by Definition 5.1 (d). Clearly, the maps

$$A \rightarrow \mathcal{C}, \quad a \mapsto 1_T \otimes_B a, \quad \text{and} \quad A^{op} \rightarrow \mathcal{C}, \quad a \mapsto a \otimes_B 1_T,$$

are algebra homomorphisms with commuting ranges in \mathcal{C} , hence \mathcal{C} is an $A \otimes_k A^{op}$ -ring. It remains to check the compatibility between its A -coring and $A \otimes_k A^{op}$ -ring structures. The Takeuchi property of the coproduct follows by Definition 5.1 (a), its multiplicativity follows by Definition 5.1 (c) and unitality follows by Definition 5.1 (d). The compatibility of the counit with the multiplication and unit is obvious. In this way \mathcal{C} is a right bialgebroid over A , and T is its right comodule algebra. By Theorem 3.4, $B \subseteq T$ is a right \mathcal{C} -Galois extension. Since T is a faithfully flat right A -module, this implies that \mathcal{C} is a right \times_A -Hopf algebra by [17, Lemma 4.1.21] (cf. Section 2).

It is proven in a symmetric way that the B -coring \mathcal{D} , associated to a faithfully flat A - B torsor T , is a left \times_B -Hopf algebra and T is its left comodule algebra.

Conversely, let T be a right faithfully flat A -ring and a left faithfully flat right \mathcal{C} -Galois extension of B , for a right \times_A -Hopf algebra \mathcal{C} . By B - A bilinearity, unitality and multiplicativity of the \mathcal{C} -coaction ϱ^T in T , $\varrho^T(ba) = b \otimes_A s(a) = \varrho^T(ab)$, where s denotes the source map in \mathcal{C} . Applying the counit we conclude that B and A are commuting subalgebras of T . Consider the pre-torsor map in Lemma 3.5. It satisfies Definition 5.1 (a) by the Takeuchi property of ϱ^T , and Definition 5.1 (b) by its right B -linearity. Definition 5.1 (d) follows by the unitality of the right \mathcal{C} -coaction ϱ^T in T , and Definition 5.1 (c) follows by its multiplicativity.

It is proven in a symmetric way that a left faithfully flat B -ring T , which is a right faithfully flat left \mathcal{D} -Galois extension of A , for a left \times_B -Hopf algebra \mathcal{D} , is a faithfully flat A - B torsor. This finishes the proof. \square

Using the monoidality of the category of comodules of a bialgebroid, one can find a simpler description of the \times_A - and \times_B -Hopf algebras, associated to a faithfully flat A - B torsor, than the one in Lemma 3.6. Similarly to [27, Theorem 3.5], they turn out to be coinvariants of diagonal comodules.

Lemma 5.3. *Let T be a faithfully flat A - B torsor with associated right \times_A -Hopf algebra \mathcal{C} and left \times_B -Hopf algebra \mathcal{D} .*

(1) View $T \otimes_A T$ as a right \mathcal{C} -comodule with the diagonal coaction

$$(5.1) \quad T \underset{A}{\otimes} T \rightarrow T \underset{A}{\otimes} T \underset{A}{\otimes} \mathcal{C}, \quad u \underset{A}{\otimes} v \mapsto u^{\langle 1 \rangle} \underset{A}{\otimes} v^{\langle 1 \rangle} \underset{A}{\otimes} v^{\langle 2 \rangle} u^{\langle 2 \rangle} \underset{B}{\otimes} u^{\langle 3 \rangle} v^{\langle 3 \rangle}.$$

Then $\mathcal{D} = (T \otimes_A T)^{co\mathcal{C}}$.

(2) View $T \otimes_B T$ as a left \mathcal{D} -comodule with the diagonal coaction

$$(5.2) \quad T \underset{B}{\otimes} T \rightarrow \mathcal{D} \underset{B}{\otimes} T \underset{B}{\otimes} T, \quad u \underset{B}{\otimes} v \mapsto u^{\langle 1 \rangle} v^{\langle 1 \rangle} \underset{A}{\otimes} v^{\langle 2 \rangle} u^{\langle 2 \rangle} \underset{B}{\otimes} u^{\langle 3 \rangle} \underset{B}{\otimes} v^{\langle 3 \rangle}.$$

Then $\mathcal{C} = {}^{co\mathcal{D}}(T \otimes_B T)$.

Proof. (1) Recall from the proof of Theorem 3.4 that an element $\sum_i u_i \otimes_A v_i \in T \otimes_A T$ belongs to \mathcal{D} if and only if

$$(5.3) \quad \sum_i u_i^{\langle 1 \rangle} \underset{A}{\otimes} u_i^{\langle 2 \rangle} \underset{B}{\otimes} u_i^{\langle 3 \rangle} v_i = \sum_i u_i \underset{A}{\otimes} v_i \underset{B}{\otimes} 1_T.$$

Application of the bijective canonical map $T \otimes_B T \rightarrow T \otimes_A \mathcal{C}$, $t \otimes_B t' \mapsto tt'^{\langle 1 \rangle} \underset{A}{\otimes} t'^{\langle 2 \rangle} \underset{B}{\otimes} t'^{\langle 3 \rangle}$ to the last two factors of (5.3) yields the equivalent condition

$$(5.4) \quad \sum_i u_i^{\langle 1 \rangle} \underset{A}{\otimes} u_i^{\langle 2 \rangle} (u_i^{\langle 3 \rangle} v_i)^{\langle 1 \rangle} \underset{A}{\otimes} (u_i^{\langle 3 \rangle} v_i)^{\langle 2 \rangle} \underset{B}{\otimes} (u_i^{\langle 3 \rangle} v_i)^{\langle 3 \rangle} = \sum_i u_i \underset{A}{\otimes} v_i 1_T^{\langle 1 \rangle} \underset{A}{\otimes} 1_T^{\langle 2 \rangle} \underset{B}{\otimes} 1_T^{\langle 3 \rangle}.$$

By Definition 5.1 (c) and (d), and Definition 3.1 (c) and (b), (5.4) is equivalent to

$$(5.5) \quad \sum_i u_i^{\langle 1 \rangle} \underset{A}{\otimes} v_i^{\langle 1 \rangle} \underset{A}{\otimes} v_i^{\langle 2 \rangle} u_i^{\langle 2 \rangle} \underset{B}{\otimes} u_i^{\langle 3 \rangle} v_i^{\langle 3 \rangle} = \sum_i u_i \underset{A}{\otimes} v_i \underset{A}{\otimes} 1_T \underset{B}{\otimes} 1_T.$$

Equation (5.5) expresses the property that $\sum_i u_i \otimes_A v_i \in T \otimes_A T$ is coinvariant with respect to the coaction (5.1). This proves claim (1). Part (2) is proven by a symmetrical reasoning. \square

Consider a right bialgebroid $\mathcal{C} = (\mathcal{C}, s, t, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$ over a k -algebra A and a k -algebra B . Write $B^e = B \otimes_k B^{op}$ for the enveloping algebra of B and view any B - B bimodule as a left B^e -module. Let T be a B^e - \mathcal{C} bicomodule. Then for every left \mathcal{C} -comodule M , the cotensor product $T \square_{\mathcal{C}} M$ inherits a left B^e -module structure of T , i.e. T induces a functor $U_T := T \square_{\mathcal{C}} \bullet : {}^{\mathcal{C}}\mathcal{M} \rightarrow {}_B\mathcal{M}_B$. Our next task is a study of this functor. Our line of reasoning follows ideas in [34], although we have to face complications caused by the fact that the algebras A and B might be different from k . Recall from [10, 39.3] that if U_T preserves colimits, then $(T \square_{\mathcal{C}} M) \otimes_R N \cong T \square_{\mathcal{C}} (M \otimes_R N)$ canonically, for any algebra R , \mathcal{C} - R bicomodule M and left R -module N .

Theorem 5.4. *Let B be an algebra, $\mathcal{C} = (\mathcal{C}, s, t, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$ a right A -bialgebroid and T a B^e - \mathcal{C} bicomodule. Let $U_T := T \square_{\mathcal{C}} \bullet : {}^{\mathcal{C}}\mathcal{M} \rightarrow {}_B\mathcal{M}_B$.*

- (1) *Every \mathcal{C} -comodule algebra structure in T which makes T a B -ring determines a lax monoidal structure of U_T .*
- (2) *If U_T preserves colimits, then every lax monoidal structure of U_T determines a \mathcal{C} -comodule algebra structure in T , such that T is a B -ring.*
- (3) *If U_T preserves colimits, then the constructions in parts (1) and (2) are mutual inverses.*

Proof. (1) The proof consists of a construction of natural homomorphisms $\xi_0 : B \rightarrow T \square_{\mathcal{C}} A$ and $\xi_{\bullet,\bullet} : (T \square_{\mathcal{C}} \bullet) \otimes_B (T \square_{\mathcal{C}} \bullet) \rightarrow T \square_{\mathcal{C}} (\bullet \otimes_{A^{op}} \bullet)$, making U_T a monoidal functor. Consider the maps

$$(5.6) \quad \xi_0(b) := 1_T b \otimes_A 1_A,$$

$$(5.7) \quad \xi_{M,M'} \left(\sum_i (u_i \otimes_A m_i) \otimes_B (u'_i \otimes_A m'_i) \right) := \sum_i u_i u'_i \otimes_A (m_i \otimes_{A^{op}} m'_i),$$

for any left \mathcal{C} -comodules M and M' . Since the cotensor product $T \square_{\mathcal{C}} M'$ is contained in the centraliser of A in the obvious A - A bimodule $T \otimes_A M'$, it is easy to see that $\xi_{M,M'}$ is well defined. By the left B -linearity and unitality of the coaction ϱ^T , and unitality of the target map t , the range of ξ_0 is in the required cotensor product. By the multiplicativity of ϱ^T , also the range of $\xi_{M,M'}$ is in the appropriate cotensor product. Obviously, both ξ_0 and $\xi_{M,M'}$ are B - B bilinear. Naturality of $\xi_{\bullet,\bullet}$ (in both arguments) follows easily by its explicit form. The hexagon identity follows by the associativity of the multiplication in T and the triangle identities follow by its unitality.

(2) The proof consists of a construction of right \mathcal{C} -colinear multiplication and unit maps, equipping T with A - and B -ring structures. Note first that the lax monoidal functor U_T maps the algebra \mathcal{C} in ${}^c\mathcal{M}$ (with unit map t and coaction $\Delta_{\mathcal{C}}$) to an algebra $T \square_{\mathcal{C}} \mathcal{C} \cong T$ in ${}_B\mathcal{M}_B$, with structure maps

$$\begin{aligned} \mu_T : (T \otimes_B T) &\xrightarrow{\varrho^T \otimes_B \varrho^T} (T \square_{\mathcal{C}} \mathcal{C}) \otimes_B (T \square_{\mathcal{C}} \mathcal{C}) \xrightarrow{\xi_{\mathcal{C},\mathcal{C}}} T \square_c (\mathcal{C} \otimes_{A^{op}} \mathcal{C}) \xrightarrow{T \square_c \mu_{\mathcal{C}}} T \square_c \mathcal{C} \xrightarrow{T \otimes_A \varepsilon_{\mathcal{C}}} T, \\ \eta_T : (B) &\xrightarrow{\xi_0} T \square_c A \xrightarrow{T \square_c t} T \square_c \mathcal{C} \xrightarrow{T \otimes_A \varepsilon_{\mathcal{C}}} T. \end{aligned}$$

This proves that T is a B -ring or, equivalently, that T is a k -algebra with unit $1_T := \eta_T(1_B)$ and $\eta_T : B \rightarrow T$ is a homomorphism of k -algebras. We make no notational difference between the multiplication maps in T as a k -algebra or B -ring. In all cases multiplication will be denoted by juxtaposition. It should be clear from the context, which structure is meant.

In order to show that T is also a \mathcal{C} -comodule algebra, we investigate properties of the map $\xi_{\mathcal{C},\mathcal{C}}$. Considering $\mathcal{C} \otimes_A \mathcal{C}$ as a left \mathcal{C} -comodule via the regular comodule structure of the first factor, the coproduct $\Delta_{\mathcal{C}}$ is left \mathcal{C} -colinear. Hence the naturality of $\xi_{\bullet,\bullet}$ implies the identities

$$(5.8) \quad (T \square_c (\mathcal{C} \otimes_{A^{op}} \Delta_{\mathcal{C}})) \circ \xi_{\mathcal{C},\mathcal{C}} = \xi_{\mathcal{C},\mathcal{C} \otimes_A \mathcal{C}} \circ ((T \square_c \mathcal{C}) \otimes_B (T \square_c \Delta_{\mathcal{C}})),$$

$$(5.9) \quad (T \square_c (\Delta_{\mathcal{C}} \otimes_{A^{op}} \mathcal{C})) \circ \xi_{\mathcal{C},\mathcal{C}} = \xi_{\mathcal{C} \otimes_A \mathcal{C},\mathcal{C}} \circ ((T \square_c \Delta_{\mathcal{C}}) \otimes_B (T \square_c \mathcal{C})).$$

For any left A -module N , $\mathcal{C} \otimes_A N$ is a left \mathcal{C} -comodule via the first factor and the map $\mathcal{C} \rightarrow \mathcal{C} \otimes_A N$, $c \mapsto c \otimes_A n$ is left \mathcal{C} -colinear, for any $n \in N$. Hence the naturality of $\xi_{\bullet,\bullet}$ implies that, for any left \mathcal{C} -comodule M , the following diagrams are commutative.

$$(5.10) \quad \begin{array}{ccc} (T \square_c M) \otimes_B (T \square_c \mathcal{C}) \otimes_A N & \xrightarrow{\cong} & (T \square_c M) \otimes_B (T \square_c (\mathcal{C} \otimes_A N)) \\ \xi_{M,\mathcal{C}} \otimes_A N \downarrow & & \downarrow \xi_{M,\mathcal{C} \otimes_A N} \\ (T \square_c (M \otimes_{A^{op}} \mathcal{C})) \otimes_A N & \xrightarrow{\cong} & T \square_c (M \otimes_{A^{op}} \mathcal{C} \otimes_A N) \end{array}$$

and

$$(5.11) \quad \begin{array}{ccc} ((T \square_c \mathcal{C}_\bullet) \otimes_B (T \square_c M)) \otimes_A N & \xrightarrow{\cong} & (T \square_c (\mathcal{C} \otimes_A N)) \otimes_B (T \square_c M) \\ \downarrow \xi_{\mathcal{C}_\bullet, M} \otimes_A N & & \downarrow \xi_{\mathcal{C} \otimes_A N, M} \\ (T \square_c (\mathcal{C}_\bullet \otimes_{A^{op}} M)) \otimes_A N & \xrightarrow{\cong} & T \square_c ((\mathcal{C}_\bullet \otimes_A N) \otimes_{A^{op}} M) \end{array}$$

The horizontal arrows are isomorphisms by the assumption that the functor $T \square_c \bullet$ preserves colimits. Put $M = N = \mathcal{C}$ in (5.10) and (5.11). Combining the resulting equalities with (5.8) and (5.9), we conclude that, for $\sum_i (u_i \otimes_A c_i) \otimes_B (u'_i \otimes_A c'_i) \in (T \square_c \mathcal{C}) \otimes_B (T \square_c \mathcal{C})$,

$$(5.12) \quad \sum_i u_i u'_i \otimes_A c_i \otimes_{A^{op}} c'_i = \xi_{\mathcal{C}, \mathcal{C}} \left(\sum_i (u_i \otimes_A c_i) \otimes_B (u'_i \otimes_A c'_i) \right),$$

proving that the left hand side is well defined. (A more detailed version of a more general computation will be presented in the proof of part (3).)

In order to check that the coaction ϱ^T is unital, introduce the notation $\sum_k u_k \otimes_A a_k := \xi_0(1_B) \in T \square_c A$. By the definition of the unit in the B -ring T , $\sum_k u_k a_k = 1_T$. With this identity at hand,

$$(5.13) \quad \varrho^T(1_T) = \sum_k u_k^{[0]} \otimes_A u_k^{[1]} s(a_k) = \sum_k u_k \otimes_A t(a_k) = \sum_k u_k a_k \otimes_A 1_{\mathcal{C}} = 1_T \otimes_A 1_{\mathcal{C}},$$

where $\varrho^T(u) = u^{[0]} \otimes_A u^{[1]}$ is the Sweedler notation for the right \mathcal{C} -coaction. The first equality follows by the right A -linearity of ϱ^T . In the second equality we used the fact that $\sum_k u_k \otimes_A a_k = \xi_0(1_B)$ belongs to the cotensor product.

Note that by the right A -linearity of the coaction ϱ^T , naturality of $\xi_{\bullet, \bullet}$ (cf. (5.10) for $N = A$), and right A -linearity of the multiplication $\mu_{\mathcal{C}}$ and counit $\varepsilon_{\mathcal{C}}$ in \mathcal{C} , the multiplication map μ_T is right A -linear. Hence, for all $a, a' \in A$,

$$\mu_T(1_{Ta} \otimes_B 1_{Ta'}) = \mu_T(1_{Ta} \otimes_B 1_T)a' = (1_{Ta})a' = 1_T(aa'),$$

where the second equality follows by the unitality of μ_T . Therefore the map

$$A \rightarrow T, \quad a \mapsto 1_{Ta}$$

is an algebra homomorphism and $ta = t(1_{Ta})$. Furthermore, by right A -linearity and unitality of ϱ^T and (5.12),

$$(1_{Ta})t = (T \otimes_A \varepsilon_{\mathcal{C}} \circ \mu_{\mathcal{C}}) \circ \xi_{\mathcal{C}, \mathcal{C}}((1_T \otimes_A s(a)) \otimes_B (t^{[0]} \otimes_A t^{[1]})) = t^{[0]} \varepsilon_{\mathcal{C}}(s(a)t^{[1]}) = at.$$

This proves that T (with the A -actions determined by its \mathcal{C} -comodule structure) is an A -ring. Furthermore, (5.12) implies, for $u, v \in T$,

$$\begin{aligned} u^{[0]} v^{[0]} \otimes_A u^{[1]} v^{[1]} &= ((T \square_c \mu_{\mathcal{C}}) \circ \xi_{\mathcal{C}, \mathcal{C}})((u^{[0]} \otimes_A u^{[1]}) \otimes_B (v^{[0]} \otimes_A v^{[1]})) \\ &= (\varrho^T \circ (T \otimes_A \varepsilon_{\mathcal{C}} \circ \mu_{\mathcal{C}}) \circ \xi_{\mathcal{C}, \mathcal{C}} \circ (\varrho^T \otimes_B \varrho^T))(u \otimes_B v) = (uv)^{[0]} \otimes_A (uv)^{[1]}. \end{aligned}$$

This proves the right \mathcal{C} -colinearity of the multiplication map in the A -ring T . The right \mathcal{C} -colinearity of the unit map is equivalent to the unitality of the coaction $\varrho^T : T \rightarrow T \times_A \mathcal{C}$, proven in (5.13). Hence T is a right \mathcal{C} -comodule algebra, as stated. This completes the proof of part (2).

(3) It is straightforward to see that starting with a right \mathcal{C} -comodule algebra T , which is also a B -ring, and applying first the construction in part (1) to it and then the one in part (2) to the result, one recovers the original algebra structure in T .

Conversely, starting with a lax monoidal structure on U_T , with coherence natural homomorphisms ξ_0 and $\xi_{\bullet,\bullet}$, applying first the construction in part (2) to it and then the one in part (1) to the result, one recovers the original natural homomorphism ξ_0 and a natural homomorphism

$$(5.14) \quad \begin{aligned} (T \square_c M) \otimes_B (T \square_c M') &\rightarrow T \square_c (M \otimes_{A^{op}} M'), \\ \sum_i (u_i \otimes_A m_i) \otimes_B (u'_i \otimes_A m'_i) &\mapsto \\ \sum_i ((T \otimes_A \varepsilon_{\mathcal{C}} \circ \mu_{\mathcal{C}}) \circ \xi_{\mathcal{C},c}) ((u_i^{[0]} \otimes_A u_i^{[1]}) \otimes_B (u'_i^{[0]} \otimes_A u'_i^{[1]})) \otimes_A (m_i \otimes_{A^{op}} m'_i) &\equiv \\ \sum_i ((T \otimes_A \varepsilon_{\mathcal{C}} \circ \mu_{\mathcal{C}}) \circ \xi_{\mathcal{C},c}) ((u_i \otimes_A m_i^{[-1]}) \otimes_B (u'_i \otimes_A m'_i^{[-1]})) \otimes_A (m_i^{[0]} \otimes_{A^{op}} m'_i^{[0]}), \end{aligned}$$

where, for all $m \in M$, ${}^M\varrho(m) = m^{[-1]} \otimes_A m^{[0]}$ is the notation for the left coaction. The claim is proven by showing that the map (5.14) is equal to $\xi_{M,M'}$. This can be seen by similar arguments to those used in proving part (2). By the left \mathcal{C} -colinearity of the left \mathcal{C} -coaction ${}^M\varrho : M \rightarrow \mathcal{C} \otimes_A M$ (where the codomain is a left \mathcal{C} -comodule via the first factor) and naturality of $\xi_{\bullet,\bullet}$,

$$(5.15) \quad (T \square_c (M \otimes_{A^{op}} {}^M\varrho)) \circ \xi_{M,M'} = \xi_{M,\mathcal{C} \otimes_A M'} \circ ((T \square_c M) \otimes_B (T \square_c {}^M\varrho))$$

$$(5.16) \quad (T \square_c ({}^M\varrho \otimes_A M')) \circ \xi_{M,M'} = \xi_{\mathcal{C} \otimes_A M, M'} \circ ((T \square_c {}^M\varrho) \otimes_B (T \square_c M')).$$

Introduce the notation $\xi_{M,M'}(\sum_i (u_i \otimes_A m_i) \otimes_B (u'_i \otimes_A m'_i)) =: \sum_j t_j \otimes_A (n_j \otimes_{A^{op}} n'_j)$, for any fixed element $\sum_i (u_i \otimes_A m_i) \otimes_B (u'_i \otimes_A m'_i) \in (T \square_{\mathcal{C}} M) \otimes_B (T \square_{\mathcal{C}} M')$. The identities (5.11), (5.16), (5.10) and (5.15) together imply the identity

$$(5.17) \quad \begin{aligned} \sum_i (\xi_{\mathcal{C},c}((u_i \otimes_A m_i^{[-1]}) \otimes_B (u'_i \otimes_A m'_i^{[-1]})) \otimes_A m_i^{[0]}) \otimes_A m'^{[0]} &= \\ \sum_j ((t_j \otimes_A (n_j^{[-1]} \otimes_{A^{op}} n_j'^{[-1]})) \otimes_A n_j^{[0]}) \otimes_A n_j'^{[0]} & \end{aligned}$$

in $((T \otimes_A (\mathcal{C} \otimes_{A^{op}} \mathcal{C})) \otimes_A M) \times_A M'$. Recall that in the tensor product $\mathcal{C} \otimes_{A^{op}} \mathcal{C}$ both module structures are given by the target map. In the tensor product $T \otimes_A (\mathcal{C} \otimes_{A^{op}} \mathcal{C})$ the left A -module structure of $\mathcal{C} \otimes_{A^{op}} \mathcal{C}$, given by right multiplication by the target map in the second factor is used. In $(T \otimes_A (\mathcal{C} \otimes_{A^{op}} \mathcal{C})) \otimes_A M$ the first tensorand $T \otimes_A (\mathcal{C} \otimes_{A^{op}} \mathcal{C})$ is understood to be a right A -module via right multiplication by the source map in the middle factor \mathcal{C} . Finally, the Takeuchi product $((T \otimes_A (\mathcal{C} \otimes_{A^{op}} \mathcal{C})) \otimes_A M) \times_A M'$ is the center of the A - A bimodule $((T \otimes_A (\mathcal{C} \otimes_{A^{op}} \mathcal{C})) \otimes_A M) \otimes_A M'$, where $((T \otimes_A (\mathcal{C} \otimes_{A^{op}} \mathcal{C})) \otimes_A M)$ is an A - A bimodule via left and right multiplications by the source map in the third factor \mathcal{C} . Consider the map

$$(5.18) \quad \begin{aligned} ((T \otimes_A (\mathcal{C} \otimes_{A^{op}} \mathcal{C})) \otimes_M M') \times_A M' &\rightarrow T \otimes_A (\mathcal{C} \otimes_{A \otimes A^{op}} \mathcal{C}) \otimes_A (M \otimes_{A^{op}} M'), \\ ((t \otimes_A (c \otimes_{A^{op}} c')) \otimes_M m) \otimes_A m' &\mapsto t \otimes_A (c \otimes_{A \otimes A^{op}} c') \otimes_A (m \otimes_{A^{op}} m'). \end{aligned}$$

In its codomain the tensor product $\mathcal{C} \otimes_{A \otimes A^{op}} \mathcal{C}$ corresponds to the $A \otimes A^{op}$ -ring structure of \mathcal{C} , via the source and target maps. In the tensor product $T \otimes_A (\mathcal{C} \otimes_{A^{op}} \mathcal{C})$ the left A -module structure of $\mathcal{C} \otimes_{A^{op}} \mathcal{C}$ is given by right multiplication by the target map in the second factor, as in the domain. In the tensor product $(\mathcal{C} \otimes_{A \otimes A^{op}} \mathcal{C}) \otimes_A (M \otimes_{A^{op}} M')$ the first factor $\mathcal{C} \otimes_{A \otimes A^{op}} \mathcal{C}$ is understood to be a right A -module via right multiplication by the source map in the second factor, and $M \otimes_{A^{op}} M'$ is a left A -module via the left A -module structure of M' . Apply the map (5.18) to both sides of (5.17) to conclude that the map (5.14) takes $\sum_i (u_i \otimes_A m_i) \otimes_B (u'_i \otimes_A m'_i) \in (T \square_{\mathcal{C}} M) \otimes_B (T \square_{\mathcal{C}} M')$ to

$$\begin{aligned} & \sum_j (T \otimes_A \varepsilon_{\mathcal{C}} \circ \mu_{\mathcal{C}})(t_j \otimes_A (n_j^{[-1]} \otimes_{A \otimes A^{op}} n'_j^{[-1]})) \otimes_A (n_j^{[0]} \otimes_{A^{op}} n'_j^{[0]}) = \\ & \sum_j t_j \otimes_A \varepsilon_{\mathcal{C}}(n_j^{[-1]} n'_j^{[-1]})(n_j^{[0]} \otimes_{A^{op}} n'_j^{[0]}) = \sum_j t_j \otimes_A \varepsilon_{\mathcal{C}}((n_j \otimes_{A^{op}} n'_j)^{[-1]})(n_j \otimes_{A^{op}} n'_j)^{[0]} = \\ & \sum_j t_j \otimes_A (n_j \otimes_{A^{op}} n'_j) = \xi_{M, M'}(u_i \otimes_A m_i) \otimes_B (u'_i \otimes_A m'_i). \end{aligned}$$

This completes the proof. \square

Lemma 5.5. *Let $\mathcal{C} = (\mathcal{C}, s, t, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$ be a right \times_A -Hopf algebra. For any left A -modules N and M , and right \mathcal{C} -comodule T , the map*

$$(5.19) \quad T \square_c ((\varepsilon_{\mathcal{C}} \otimes N) \otimes_{A^{op}} (\mathcal{C} \otimes M)) : T \square_c ((\mathcal{C} \otimes N) \otimes_{A^{op}} (\mathcal{C} \otimes M)) \rightarrow (T \otimes_A \mathcal{C}_\bullet \otimes N) \otimes_A M,$$

is a bijection. In the domain of (5.19), \mathcal{C} is a right A -module through the right multiplication by the source map, $\mathcal{C} \otimes_A M$ and $\mathcal{C} \otimes_A N$ are left \mathcal{C} -comodules via the regular comodule structure of their first factors, and $(\mathcal{C} \otimes_A N) \otimes_{A^{op}} (\mathcal{C} \otimes_A M)$ is a left \mathcal{C} -comodule via the diagonal coaction. In the codomain, \mathcal{C} in $\mathcal{C} \otimes_A N$ is the right A -module through the left multiplication by the target map, while in $\mathcal{C} \otimes_A M$ it is a right A -module through the right multiplication by the source map. In $T \otimes_A \mathcal{C}$ the left A -module structure of \mathcal{C} is understood via right multiplication by the target map.

Proof. The map (5.19) is well defined by the A - A bilinearity of $\varepsilon_{\mathcal{C}}$. Its bijectivity is proven by constructing the inverse in terms of the inverse of the Galois map $\theta : \mathcal{C} \otimes_{A^{op}} \mathcal{C} \rightarrow \mathcal{C} \otimes_A \mathcal{C}$, $c \otimes_A c' \mapsto c \Delta_{\mathcal{C}}(c')$. Consider the map

$$\begin{aligned} & ((T \otimes_A \mu_{\mathcal{C}} \otimes_{A^{op}} \mathcal{C}) \circ (\varrho^T \otimes_A \theta^{-1}(1_{\mathcal{C}} \otimes_A \bullet)) \otimes_N) \otimes_M : \\ & (T \otimes_A \mathcal{C}_\bullet \otimes N) \otimes_M \rightarrow ((T \otimes_A \mathcal{C}_\bullet \otimes \mathcal{C}_\circ) \otimes_A N) \otimes_M \cong T \otimes_A ((\mathcal{C} \otimes_N) \otimes_{A^{op}} (\mathcal{C} \otimes_M)). \end{aligned}$$

It is well defined by the multiplicativity of the translation map $\theta^{-1}(1_{\mathcal{C}} \otimes_A \bullet)$ and (2.4). Application of the pentagon identity (2.3) in the form

$$\theta_{13}^{-1} \circ (\mathcal{C} \otimes_A \theta^{-1}) \circ (\theta \otimes_A \mathcal{C}) = (\theta \otimes_{A^{op}} \mathcal{C}) \circ (\mathcal{C} \otimes_{A^{op}} \theta^{-1})$$

yields that the range of (5.20) is in the cotensor product $T \square_{\mathcal{C}} ((\mathcal{C} \otimes_A N) \otimes_{A^{op}} (\mathcal{C} \otimes_A M))$. We leave it to the reader to check that the maps (5.19) and (5.20) are mutual inverses. \square

Theorem 5.6. *Let $\mathcal{C} = (\mathcal{C}, s, t, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$ be a right \times_A -Hopf algebra. Let T be a right \mathcal{C} -comodule algebra and B a subalgebra of $T^{co\mathcal{C}}$. Denote by $U_T := T \square_{\mathcal{C}} \bullet$ the induced lax monoidal functor $\mathcal{C}\mathcal{M} \rightarrow B\mathcal{M}_B$.*

- (1) If U_T is a monoidal functor, then $B \subseteq T$ is a right \mathcal{C} -Galois extension.
- (2) If $B \subseteq T$ is a right faithfully flat right \mathcal{C} -Galois extension, then U_T is a monoidal functor.

Proof. (1) Via the embedding $T \square_{\mathcal{C}} A \hookrightarrow T \otimes_A A \cong T$, the coinvariants of the right \mathcal{C} -comodule T are identified with the elements of $T \square_{\mathcal{C}} A$. By the monoidality of U_T , $\xi_0 : B \rightarrow T \square_{\mathcal{C}} A$ is an isomorphism. Hence B is equal to $T^{co\mathcal{C}}$.

In light of the form (5.7) of the natural homomorphism $\xi_{\bullet,\bullet} : (T \square_{\mathcal{C}} \bullet) \otimes_B (T \square_{\mathcal{C}} \bullet) \rightarrow T \square_{\mathcal{C}} (\bullet \otimes_{A^{op}} \bullet)$, the canonical map $T \otimes_B T \rightarrow T \otimes_A \mathcal{C}$ can be written as a composite map,

$$(5.21) \quad T \otimes_B T \xrightarrow{\varrho^T \otimes_B \varrho^T} (T \square_c \mathcal{C}) \otimes_B (T \square_c \mathcal{C}) \xrightarrow{\xi_{c,c}} T \square_c (\mathcal{C} \otimes_{A^{op}} \mathcal{C}) \xrightarrow{T \square_c (\varepsilon_{\mathcal{C}} \otimes_{A^{op}} \mathcal{C})} T \otimes_A \mathcal{C}.$$

Since $\varrho^T : T \rightarrow T \square_{\mathcal{C}} \mathcal{C}$ is an isomorphism, so is the first arrow in (5.21). The middle arrow is an isomorphism by the monoidality of U_T . The last arrow is an isomorphism by Lemma 5.5. This proves that the canonical map (5.21) is bijective, that is, $B \subseteq T$ is a right \mathcal{C} -Galois extension.

(2) We need to show that the natural homomorphisms (5.6) and (5.7) are isomorphisms. Since the canonical map $\text{can} : T \otimes_B T \rightarrow T \otimes_A \mathcal{C}$ is an isomorphism by assumption, so is the map

$$\gamma_M : T \otimes_B (T \square_c M) \cong (T \otimes_B T) \square_c M \xrightarrow{\text{can} \square_c M} (T \otimes_A \mathcal{C}) \square_c M \cong T \otimes_A M ,$$

mapping $\sum_i u_i \otimes_B v_i \otimes_A m_i$ to $\sum_i u_i v_i \otimes_A m_i$, for any left \mathcal{C} -comodule M . Since $T \square_c M$ is contained in the centraliser of A in the obvious A - A bimodule $T \otimes_A M$, the map γ_M is right A -linear with respect to the right A -module structure of $T \otimes_B (T \square_c M)$, via its first factor. A straightforward computation shows the commutativity of the following diagram, for any left \mathcal{C} -comodules M and M' ,

$$\begin{array}{ccc} T \otimes_B (T \square_c M) \otimes_B (T \square_c M') & \xrightarrow{\gamma_M \otimes_B (T \square_c M')} & (T \otimes_A M) \otimes_B (T \square_c M') \\ \downarrow T \otimes_B \xi_{M,M'} & & \downarrow \gamma_{M'} \otimes_A M \\ T \otimes_B (T \square_c (M \otimes_{A^{op}} M')) & \xrightarrow{\gamma_{M \otimes_{A^{op}} M'}} & T \otimes_A M' \otimes_A M \\ & & \cong T \otimes_A (M \otimes_{A^{op}} \bullet M') \end{array}$$

Therefore $T \otimes_B \xi_{M,M'}$ is an isomorphism. Since T is a faithfully flat right B -module by assumption, this proves that $\xi_{M,M'}$ is an isomorphism, for any left \mathcal{C} -comodules M and M' . By one of the triangle identities, $\xi_{c,A} \circ (\varrho^T \otimes_B (T \square_c A)) \circ (T \otimes_B \xi_0)$ is an isomorphism. Hence so is $T \otimes_B \xi_0$, and, by the faithful flatness of the right B -module T , also ξ_0 . This completes the proof. \square

In particular Theorem 5.6 implies that a faithfully flat A - B torsor which is also faithfully flat as a right B -module induces a k -linear monoidal functor ${}^C\mathcal{M} \rightarrow {}_B\mathcal{M}_B$. In the case when A and B are both equal to the ground ring k (and hence the faithful flatness assumptions made on T imply properties (iv) in Remark 4.7), the induced functor is known to be a ‘fibre functor’. That is, it is faithful and preserves colimits and kernels. Actually it is not hard to see that also for arbitrary k -algebras A and B , properties (iv) in Remark 4.7 imply these properties of the induced functor.

Furthermore, in the case when A and B are trivial, every fibre functor is known to be induced by a faithfully flat torsor. In our general setting it follows by Theorems 5.4 and 5.6 that, for a right \times_A -Hopf algebra \mathcal{C} , an algebra B , and a k -linear faithful monoidal functor $U : {}^c\mathcal{M} \rightarrow {}_B\mathcal{M}_B$, preserving colimits and kernels, $U(\mathcal{C})$ is a right \mathcal{C} -Galois extension of B . However, we were not able to derive properties (iv) in Remark 4.7 of $U(\mathcal{C})$.

Finally, we show that the functor induced by a torsor satisfying property (iii) in Remark 4.7 is a monoidal equivalence.

Theorem 5.7. *Let T be an A - B torsor which obeys properties (iii) in Remark 4.7. Then the functor $T\square_{\mathcal{C}}\bullet : {}^c\mathcal{M} \rightarrow {}^{\mathcal{D}}\mathcal{M}$ is a k -linear monoidal equivalence between the categories of comodules for the associated right \times_A -Hopf algebra \mathcal{C} and left \times_B -Hopf algebra \mathcal{D} .*

Proof. The functor $T\square_{\mathcal{C}}\bullet : {}^c\mathcal{M} \rightarrow {}^{\mathcal{D}}\mathcal{M}$ is a k -linear equivalence by Corollary 4.8. The functor $T\square_{\mathcal{C}}\bullet : {}^c\mathcal{M} \rightarrow {}_B\mathcal{M}_B$ is monoidal by Theorem 5.6 (2). Recall the existence of a strict monoidal forgetful functor ${}^{\mathcal{D}}\mathcal{M} \rightarrow {}_B\mathcal{M}_B$. It is straightforward to see that the left \mathcal{D} -comodule algebra structure of T implies the left \mathcal{D} -colinearity of the coherence natural isomorphisms (5.6) and (5.7). This completes the proof. \square

Note that in the case when the algebras A and B are equal to the ground ring k , also a converse of Theorem 5.7 holds: every k -linear monoidal equivalence between comodule categories of flat Hopf algebras is induced by a faithfully flat torsor [27, Corollary 5.7]. In our general setting, however, by the failure of properties (iii) in Remark 4.7 of $F(\mathcal{C})$, for a k -linear monoidal equivalence functor $F : {}^c\mathcal{M} \rightarrow {}^{\mathcal{D}}\mathcal{M}$, we were not able to prove an analogous result.

APPENDIX A. CLEFT EXTENSIONS BY HOPF ALGEBROIDS

A Hopf algebroid consists of a compatible pair of a right and a left bialgebroid – on the same total algebra over a commutative ring k but over anti-isomorphic base k -algebras A and L . In addition there is an antipode map. References about Hopf algebroids are [6] (in the case when the antipode is bijective) and [3] (in general). As a consequence of the compatibility of the two involved bialgebroid structures, the categories of their (right, say) comodules are isomorphic monoidal categories. The isomorphism is compatible with the forgetful functors to the category of k -modules. Its explicit form was given in [5, Theorem 2.2].

Since in a Hopf algebroid there are two bialgebroid (and hence coring) structures present, it is helpful to use two versions of Sweedler's index notation. Upper indices are used to denote the components of the coproduct in the right bialgebroid, i.e. we write $h \mapsto h^{(1)} \otimes_A h^{(2)}$. Lower indices, $h \mapsto h_{(1)} \otimes_L h_{(2)}$, denote the coproduct of left bialgebroid. Similar upper/lower index conventions are used for right coactions of the right/left bialgebroid (related by the isomorphism in [5, Theorem 2.2]), i.e. notations $t \mapsto t^{[0]} \otimes_A t^{[1]} / t \mapsto t_{[0]} \otimes_L t_{[1]}$ are used.

Cleft extensions by Hopf algebroids were introduced in [5]. In [5, Theorem 3.12] these extensions were characterised as Galois extensions by the constituent right bialgebroid that satisfy normal basis property with respect to the constituent left bialgebroid.

Consider a Hopf algebroid \mathcal{H} , with constituent right bialgebroid over an algebra A and left bialgebroid over L . Let T be a right faithfully flat A -ring and a left faithfully

flat right *cleft* extension of B by \mathcal{H} . In this situation, by Theorem 5.2 and Lemma 5.3, there exists a canonical left \times_B -Hopf algebra structure in $(T \otimes_A T)^{co\mathcal{H}}$. The aim of this section is to determine it explicitly. This results in new examples of bialgebroids which generalise the extended Hopf algebra of Connes and Moscovici introduced in [11] in the context of transverse geometry. We start with the following generalisation of a construction in [18] (see also [24]). For definitions of an algebra measured by a bialgebroid, an (invertible) 2-cocycle on a bialgebroid, a cocycle twisted module of a bialgebroid, a crossed product by a bialgebroid and a cleft extension by a Hopf algebroid, we refer to [5].

Proposition A.1. *Let \mathcal{H} be a left \times_L -Hopf algebra, and B an L -ring measured by \mathcal{H} , with measuring \cdot . Let σ be a B -valued 2-cocycle on \mathcal{H} , that makes B a σ -twisted \mathcal{H} -module. Consider the k -module $\mathcal{D} := B \otimes_L (B \otimes_L \bullet \mathcal{H})$, where the L -module tensor product in the parenthesis is understood with respect to the left L -module structure of \mathcal{H} , given through the right multiplication by the target map, and the resulting tensor product is meant to be a left L -module via left multiplication by the source map in the second factor \mathcal{H} . It has the following structures.*

- (1) \mathcal{D} is a $B \otimes_k B^{op}$ -ring, with source and target maps

$$s_{\mathcal{D}}(b) := b \underset{L}{\otimes} 1_B \underset{L}{\otimes} 1_{\mathcal{H}} \quad \text{and} \quad t_{\mathcal{D}}(b) := 1_B \underset{L}{\otimes} b \underset{L}{\otimes} 1_{\mathcal{H}},$$

respectively, and multiplication

$$(b \underset{L}{\otimes} b' \underset{L}{\otimes} h) (c \underset{L}{\otimes} c' \underset{L}{\otimes} k) := b(h^{(1)}_{(1)} \cdot c) \sigma(h^{(1)}_{(2)}, k^{(1)}_{(1)}) \underset{L}{\otimes} c'(k^{(2)}_{(1)} \cdot b') \sigma(k^{(2)}_{(2)}, h^{(2)}) \underset{L}{\otimes} h^{(1)}_{(3)} k^{(1)}_{(2)},$$

where the notation, $\theta^{-1}(h \otimes_L 1_{\mathcal{H}}) = h^{(1)} \otimes_{L^{op}} h^{(2)}$ (with implicit summation) is used for the inverse of the Galois map $\theta(h \otimes_{L^{op}} k) = \Delta_{\mathcal{H}}(h)k$, for $h, k \in \mathcal{H}$.

- (2) If σ is an invertible cocycle (with inverse $\tilde{\sigma}$), then \mathcal{D} is a left B -bialgebroid with coproduct and counit

$$\begin{aligned} \Delta_{\mathcal{D}}(b \underset{L}{\otimes} b' \underset{L}{\otimes} h) &= (b \underset{L}{\otimes} \tilde{\sigma}(h_{(1)}^{(2)}, h_{(2)}) \underset{L}{\otimes} h_{(1)}^{(1)}) \underset{B}{\otimes} (1_B \underset{L}{\otimes} b' \underset{L}{\otimes} h_{(3)}) \quad \text{and} \\ \varepsilon_{\mathcal{D}}(b \underset{L}{\otimes} b' \underset{L}{\otimes} h) &= b(h_{(1)} \cdot b') \sigma(h_{(2)}^{(1)}, h_{(2)}^{(2)}). \end{aligned}$$

- (3) If σ is an invertible cocycle (with inverse $\tilde{\sigma}$), then \mathcal{D} is a left \times_B -Hopf algebra. That is, the Galois map $\mathcal{D} \otimes_{B^{op}} \mathcal{D} \rightarrow \mathcal{D} \otimes_B \mathcal{D}$, $(b \otimes_L b' \otimes_L h) \otimes_{B^{op}} (c \otimes_L c' \otimes_L k) \mapsto (\Delta_{\mathcal{D}}(b \otimes_L b' \otimes_L h))(c \otimes_L c' \otimes_L k)$ is bijective, with inverse

$$\begin{aligned} (b \underset{L}{\otimes} b' \underset{L}{\otimes} h) \underset{B}{\otimes} (c \underset{L}{\otimes} c' \underset{L}{\otimes} k) &\mapsto (b \underset{L}{\otimes} 1_B \underset{L}{\otimes} h^{(1)}) \underset{B^{op}}{\otimes} (b'(h^{(2)}_{(1)} \cdot c) \sigma(h^{(2)}_{(2)}, k^{(1)}_{(1)}) \\ &\quad \underset{L}{\otimes} c' \tilde{\sigma}(k^{(2)} h^{(2)}_{(3)} \underset{L}{\otimes} h^{(2)}_{(4)}, h^{(2)}_{(3)} \underset{L}{\otimes} h^{(2)}_{(3)} \underset{L}{\otimes} k^{(1)}_{(2)})). \end{aligned}$$

Verification of Proposition A.1 by a direct (and somewhat long) computation is left to the reader.

Remark A.2. (1) Proposition A.1 can be specialised to the case when \mathcal{H} is a Hopf algebra, and B is its module algebra (i.e. σ is a trivial cocycle). Then the \times_B -Hopf algebra \mathcal{D} in the proposition reduces to the one constructed in [18], which, subsequently has been shown in [24] to be isomorphic to the bialgebroid constructed in [11, Section 3].

(2) Apply Proposition A.1 to the particular case when the σ -twisted \mathcal{H} -module B is equal to the base algebra L itself. Then \mathcal{D} is isomorphic to \mathcal{H} , as a k -module. If σ is an

invertible cocycle, then the k -linear automorphism of \mathcal{H} , $h \mapsto t(\sigma(h_{(2)}^{(1)}, h_{(2)}^{(2)}))h_{(1)}$, with inverse $h \mapsto h_{(1)}^{(1)}t(\tilde{\sigma}(h_{(1)}^{(2)}, h_{(2)}))$, maps the left \times_B -Hopf algebra \mathcal{D} to the *cocycle double twist* of \mathcal{H} . By a cocycle double twist of a left bialgebroid we mean the following generalisation of Doi's construction on bialgebras in [12]. Let \mathcal{H} be a left L -bialgebroid, with source map s and target map t . Assume that \mathcal{H} measures L and let σ be an L -valued invertible 2-cocycle. Then \mathcal{H} is a left L -bialgebroid with unmodified source and target maps and L -coring structure, and a newly defined product

$$h \otimes_k h' \mapsto s(\sigma(h_{(1)}, h'_{(1)}))t(\tilde{\sigma}(h_{(3)}, h'_{(3)}))h_{(2)}h'_{(2)}, \quad \text{for } h, h' \in \mathcal{H}.$$

Theorem A.3. *Let \mathcal{H} be a Hopf algebroid, with a constituent left L -bialgebroid, right A -bialgebroid, and antipode S . Let T be a right faithfully flat A -ring and a left faithfully flat right \mathcal{H} -cleft extension of B . Then the canonical left \times_B -Hopf algebra in Theorem 5.2 (iii) is isomorphic to \mathcal{D} in Proposition A.1.*

Proof. Let $j : \mathcal{H} \rightarrow T$ be a normalised cleaving map, with convolution inverse \tilde{j} . Recall from [5, Theorem 3.12] that T is isomorphic to $B \otimes_L \mathcal{H}$, as a left B -module and right \mathcal{H} -comodule. Hence a k -linear isomorphism $(T \otimes_A T)^{co\mathcal{H}} \rightarrow \mathcal{D}$ is given by the map

$$(A.1) \quad u \otimes_A v \mapsto u_{[0]}^{[0]} \tilde{j}(u_{[0]}^{[1]}) \otimes_L v^{[0]} \tilde{j}(v^{[1]}) \otimes_L u_{[1]},$$

with inverse $b \otimes_L b' \otimes_L h \mapsto b j(h^{(1)}) \otimes_A b' j(S(h^{(2)}))$. Recall that the constituent left L -bialgebroid in a Hopf algebroid is a \times_L -Hopf algebra, with inverse Galois map $\theta^{-1}(h \otimes_L k) = h^{(1)} \otimes_{L^{op}} S(h^{(2)})k$, for $h \otimes_L k \in \mathcal{H} \otimes_L \mathcal{H}$. Using the explicit form $u \mapsto u_{[0]}^{[0]} \otimes_A \tilde{j}(u_{[0]}^{[1]}) \otimes_B j(u_{[1]})$ of the torsor map, for $u \in T$, and relations [5, (4.18) and (4.19)] between cleaving maps and 2-cocycles, the reader can easily check that the isomorphism (A.1) preserves the B -bialgebroid structure indeed. \square

APPENDIX B. PRE-TORSORS AS DIFFERENTIABLE BIMODULES

In this appendix we calculate explicitly differential graded algebras arising from pre-torsors and describe differentiable bimodule structures on a pre-torsor. Such bimodules are one of the main ingredients in the construction of non-commutative differential fibrations [2]. Recall that, given a differential graded algebra $(\Omega(A), d)$ with $\Omega^0(A) = A$ and a right A -module M , a map $\nabla : M \rightarrow M \otimes_A \Omega^1(A)$ is called a (right) *connection* in M , provided that for all $a \in A$ and $m \in M$,

$$\nabla(ma) = \nabla(m)a + m \otimes_A da.$$

Similarly, one defines a connection in a left A -module. The map ∇ can be uniquely extended to the map $\nabla : M \otimes_A \Omega^\bullet(A) \rightarrow M \otimes_A \Omega^{\bullet+1}(A)$ by the (graded in the left module case) Leibniz rule: $\nabla(m \otimes_A \omega) = \nabla(m)\omega + m \otimes_A d\omega$. A connection ∇ is said to be *flat*, provided $\nabla \circ \nabla = 0$.

We say that an A - B pre-torsor T is *unital* if the structure map τ satisfies condition (d) in Definition 5.1. For example, the pre-torsor constructed in Example 3.9 is unital. If, in addition, T is a faithfully flat pre-torsor, then A -coring \mathcal{C} in Theorem 3.4 has a group-like element $1_T \otimes_B 1_T$. Now, [10, 29.7, 29.14] yield the following corollary of Theorem 3.4.

Corollary B.1. *Let T be a faithfully flat unital A - B pre-torsor with the torsor map τ . Then*

- (1) (a) There exists a differential graded (tensor) algebra structure $\Omega(A)$ on A with

$$\Omega^1(A) := \left\{ \sum_i t_i \otimes_B u_i \in T \otimes_B T \mid \sum_i t_i u_i = 0 \text{ and } \sum_i t_i \tau(u_i) = \sum_i 1_T \otimes_A t_i \otimes_B u_i \right\},$$

$\Omega^n(A) = \Omega^1(A)^{\otimes_A n}$, and differentials $d(a) = 1_T \otimes_B \alpha(a) - \alpha(a) \otimes_B 1_T$, for all $a \in A$ and, for all $\sum_i t_i \otimes_B u_i \in \Omega^1(A)$,

$$d\left(\sum_i t_i \otimes_B u_i\right) = \sum_i 1_T \otimes_B 1_T \otimes_A t_i \otimes_B u_i - \sum_i t_i \otimes_B \tau(u_i) + \sum_i t_i \otimes_B u_i \otimes_A 1_T \otimes_B 1_T.$$

- (b) The map

$$\nabla_T^r : T \rightarrow T \otimes_A \Omega^1(A), \quad t \mapsto \tau(t) - t \otimes_A 1_T \otimes_B 1_T,$$

is a flat connection in T .

- (2) (a) There exists a differential graded (tensor) algebra structure $\Omega(B)$ on B with

$$\Omega^1(B) := \left\{ \sum_i t_i \otimes_A u_i \in T \otimes_A T \mid \sum_i t_i u_i = 0 \text{ and } \sum_i \tau(t_i) u_i = \sum_i t_i \otimes_A u_i \otimes_B 1_T \right\},$$

$\Omega^n(B) = \Omega^1(B)^{\otimes_B n}$, and differentials $d(b) = 1_T \otimes_A \beta(b) - \beta(b) \otimes_A 1_T$, for all $b \in B$ and, for all $\sum_i t_i \otimes_A u_i \in \Omega^1(B)$,

$$d\left(\sum_i t_i \otimes_A u_i\right) = \sum_i 1_T \otimes_A 1_T \otimes_B t_i \otimes_A u_i - \sum_i \tau(t_i) \otimes_A u_i + \sum_i t_i \otimes_A u_i \otimes_B 1_T \otimes_A 1_T.$$

- (b) The map

$$\nabla_T^l : T \rightarrow \Omega^1(B) \otimes_B T, \quad t \mapsto 1_T \otimes_A 1_T \otimes_B t - \tau(t),$$

is a flat connection in T .

Recall from [2, Definition 2.10] (cf. [22, Section 3.6]) that, given differential graded algebras $\Omega(A)$ over A and $\Omega(B)$ over B , a B - A -bimodule M with a left connection $\nabla : M \rightarrow \Omega^1(B) \otimes_B M$ and a B - A bimodule map $\sigma : M \otimes_A \Omega^1(A) \rightarrow \Omega^1(B) \otimes_B M$ is called a (*left*) *differentiable bimodule*, provided that, for all $m \in M$ and $a \in A$, $\nabla(ma) = \nabla(m)a + \sigma(m \otimes_A da)$. In case $A = B$, the pair (∇, σ) is termed a (*left*) *A -bimodule connection* [14]. Differentiable bimodules induce functors between categories of connections (cf. [2, Proposition 2.12]). Since the left connection ∇_T^l in Corollary B.1(2)(b) is right A -linear, every faithfully flat unital A - B torsor is a differentiable bimodule with the trivial (zero) map σ . The corresponding functor coincides with the tensor (induction) functor $T \otimes_A \bullet$. More interesting differential bimodule structure can be constructed on T , provided the structure map τ is B - B bilinear.

Proposition B.2. *Let T be a faithfully flat unital A - B pre-torsor with differential structures $\Omega(A)$ and $\Omega(B)$ as in Corollary B.1.*

- (1) The pair (∇_T^l, σ_B) , where

$$\sigma_B : T \otimes_B \Omega^1(B) \rightarrow \Omega^1(B) \otimes_B T, \quad \sum_i t \otimes_B u_i \otimes_A v_i \mapsto \sum_i \tau(tu_i)v_i.$$

is a B -bimodule connection.

- (2) If the torsor structure map τ is right B -linear, then T is a differentiable B - A bimodule with (flat) connection ∇_T^l in Corollary B.1(2)(b) and the twist map

$$\sigma^l : T \otimes_A \Omega^1(A) \rightarrow \Omega^1(B) \otimes_B T, \quad \sum_i t \otimes u_i \otimes v_i \mapsto \sum_i \tau(tu_i)v_i.$$

Proof. (1) First we need to check whether the map σ_B is well-defined. The property (a) in Definition 3.1 implies that $\text{Im} \sigma_B \subseteq \ker(\mu_T \otimes_B T)$. Second, by properties (c) and (a) in Definition 3.1,

$$(T \otimes_A T \otimes_B \mu_T \otimes_B T) \circ (\tau \otimes_A T \otimes_B T) \circ \sigma_B = (T \otimes_A T \otimes_B \beta \otimes_B T) \circ \sigma_B.$$

Since T is a (faithfully) flat left B -module, this implies that the map σ_B is well-defined. An easy calculation verifies the twisted Leibniz rule.

(2) The map σ^l is well-defined by the same arguments as in part (1) and the right B -linearity of τ . Since ∇_T^l is a right A -module map, we only need to show that, for all $a \in A$ and $t \in T$, $\sigma^l(t \otimes_A da) = 0$. This follows by the right A -linearity of τ . \square

Note that the twisting map σ_B in Proposition B.2 is a restriction of $\psi_{\mathcal{D}}$ in (4.2).

ACKNOWLEDGEMENTS

The work of the first author is supported by the Hungarian Scientific Research Fund OTKA T 043 159 and the Bolyai János Fellowship. She thanks C. Menini for useful discussions. The second author would like to express his gratitude to the members of the Department of Theoretical Physics at the Research Institute for Particle and Nuclear Physics in Budapest for very warm hospitality.

REFERENCES

- [1] J. Abuhlail, *Morita contexts for corings and equivalences*, in S. Caenepeel and F. Oystaeyen (eds.) Lecture Notes in Pure and Applied Math. Vol. 239 Marcel Dekker New York 2005, pp 1–19.
- [2] E.J. Beggs and T. Brzeziński, *The Serre spectral sequence of a noncommutative fibration for de Rham cohomology*, Acta Math. 195 (2005), 155–196.
- [3] G. Böhm, *Integral theory for Hopf algebroids*, Alg. Rep. Theory 8(4) (2005), 563–599.
- [4] G. Böhm and T. Brzeziński, *Strong connections and the relative Chern-Galois character for corings*, Int. Math. Res. Notices, 2005:42 (2005), 2579–2625.
- [5] G. Böhm and T. Brzeziński, *Cleft extensions of Hopf algebroids*, Appl. Cat. Str. 14 (2006), 431–469.
- [6] G. Böhm and K. Szlachányi, *Hopf algebroids with bijective antipodes: axioms, integrals and duals*, J. Algebra 274 (2004), 585–617.
- [7] G. Böhm and J. Vercruyse, *Morita theory for coring extensions and cleft bicomodules*, Adv. Math. 209 (2007), 611–648.
- [8] T. Brzeziński, *A note on coring extensions*, Annali dell’Università di Ferrara, Sezione VII.-Sci. Mat. 51 (2005), 15–27.
- [9] T. Brzeziński and P.M. Hajac, *Coalgebra extensions and algebra coextensions of Galois type*, Comm. Alg. 27 (1999), 1347–1367.
- [10] T. Brzeziński and R. Wisbauer, *Corings and Comodules*. Cambridge University Press, Cambridge, 2003. Erratum: <http://www-maths.swan.ac.uk/staff/tb/corinerr.pdf>
- [11] A. Connes and H. Moscovici, *Rankin-Cohen brackets and the Hopf algebra of transverse geometry*, Mosc. Math. J. 4 (2004), 111–130.
- [12] Y. Doi, *Braided bialgebras and quadratic bialgebras*, Comm. Algebra 21 (1993), 1731–1749.
- [13] P. Deligne and J. Milne, *Tannakian Categories*, Lecture Notes Math. 900, Springer 1982, pp 101–228.

- [14] M. Dubois-Violette, J. Madore, T. Masson, J. Mourad, *On curvature in noncommutative geometry*, J. Math. Phys. 37 (1996), 4089–4102.
- [15] L. El Kaoutit, J. Gómez-Torrecillas and F.J. Lobillo, *Semisimple corings*, Algebra Colloquium 11:4 (2004), 427–442.
- [16] C. Grunspan, *Quantum torsors*, J. Pure Appl. Alg. 184 (2003), 229–255.
- [17] D. Hobst, *Antipodes in the theory of noncommutative torsors*, PhD thesis Ludwig-Maximilians Universität München (2004), Logos Verlag Berlin, 2004.
- [18] L. Kadison, *Pseudo-Galois extensions and Hopf algebroids*, arXiv:math.QA/0508411.
- [19] L. Kadison and K. Szlachányi, *Bialgebroid actions on depth two extensions and duality*, Adv. Math. 179 (2003), 75–121.
- [20] M. Kontsevich, *Operads and motives in deformation quantisation*, Lett. Math. Phys. 48 (1999), 35–72.
- [21] J.-H. Lu, *Hopf algebroids and quantum groupoids*, Int. J. Math. 7 (1996) 47–70.
- [22] J. Madore, *An Introduction to Noncommutative Differential Geometry and Its Physical Applications*. 2nd ed., Cambridge University Press, Cambridge (1999).
- [23] I. Moerdijk and J. Mrčun, *Introduction to Foliations and Lie Groupoids*, Cambridge University Press, Cambridge (2003).
- [24] F. Panaite and F. Van Oystaeyen, *Some bialgebroids constructed by Kadison and Connes-Moscovici are isomorphic*, Appl. Cat. Str. 14 (2006), 627–632.
- [25] E.F. Müller and H.-J. Schneider, Quantum homogenous spaces with faithfully flat module structures, Israel J. Math. 111 (1999), 157–190.
- [26] N. Saavedra Rivano, *Catégories Tannakiennes* Lecture Notes in Math. 265, Springer 1972.
- [27] P. Schauenburg, *Hopf bi-Galois extensions*, Commun. Algebra 24 (1996), 3797–3825.
- [28] P. Schauenburg, *Bialgebras over noncommutative rings and a structure theorem for Hopf bimodules*, Appl. Cat. Str. 6 (1998), 193–222.
- [29] P. Schauenburg, *Duals and doubles of quantum groupoids (\times_R -Hopf algebras)*, in N. Andruskiewitsch, W.R. Ferrer Santos, H.-J. Schneider (eds.) *New trends in Hopf algebra theory*, Contemp. Math. vol. 267, Amer. Math. Soc., Providence, RI, (2000), pp. 273–299.
- [30] P. Schauenburg, *Quantum torsors with fewer axioms*, arXiv: math.QA/0302003.
- [31] P. Schauenburg, *Hopf-Galois and bi-Galois extensions*, in *Galois theory, Hopf algebras, and semiabelian categories*, Fields Inst. Commun., 43, Amer. Math. Soc., Providence, RI, (2004), pp. 469–515.
- [32] H.-J. Schneider, Principal homogenous spaces for arbitrary Hopf algebras, Israel J. Math. 72 (1990), 167–195.
- [33] M. Takeuchi, *Groups of algebras over $A \otimes \bar{A}$* , J. Math. Soc. Japan 29 (1977), 459–492.
- [34] K.-H. Ulbrich, *Galois extensions as functors of comodules*, Manuscripta Math. 59 (1987), 391–397.
- [35] K.-H. Ulbrich, *Fibre functors of finite dimensional comodules*, Manuscripta Math. 65 (1989), 39–46.
- [36] K.-H. Ulbrich, *On Hopf algebras and rigid monoidal categories*, Israel J. Math. 72 (1990), 252–256.

RESEARCH INSTITUTE FOR PARTICLE AND NUCLEAR PHYSICS, BUDAPEST,
H-1525 BUDAPEST 114, P.O.B. 49, HUNGARY
E-mail address: G.Bohm@rmki.kfki.hu

DEPARTMENT OF MATHEMATICS, SWANSEA UNIVERSITY, SINGLETON PARK,
SWANSEA SA2 8PP, U.K.
E-mail address: T.Brzezinski@swansea.ac.uk